Entropy Methods for Reaction-Diffusion Equations: Degenerate Diffusion and Slowly Growing A-priori Bounds.

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Abstract

In the continuation of [DF], we study reversible reaction-diffusion equations via entropy methods (based on the free energy functional) in two situations of degeneracy: Firstly, for a two species system, we show explicit exponential convergence to the unique constant steady state when spatial diffusion of one specie vanishes but the system still obeys the same steady state. Secondly, for a system of four species in 1D, we deduce 1) an at most polynominally growing L^∞-bound from a-priori-estimates on the entropy and entropy dissipation, 2) almost exponential convergence to the steady state via a precise entropy-entropy dissipation estimate, 3) an explicit global L^∞-bound via interpolation of a polynominally growing H^1-bound with the almost exponential L^1-convergence, and 4), finally, explicit exponential convergence to the steady state in all Sobolev norms.

Key words: Reaction-Diffusion, Entropy Method, Exponential Decay, Degenerate Diffusion, Slowly Growing A-Priori-Estimates

AMS subject classification: 35B40, 35K57

Acknowledgement: This work has been supported by the European IHP network "HYKE-HYperbolic and Kinetic Equations: Asymptotics, Numerics, Analysis", Contract Number: HPRN-CT-2002-00282. K.F. has also been supported by then Austrian Science Fund FWF project P16174-N05 and by the Wittgenstein Award of P. A. Markowich.

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1 Introduction

1.1 General presentation

We study the large-time behaviour of diffusive and reversible chemical reactions of the type

$$\alpha_1 \mathcal{A}_1 + \ldots + \alpha_q \mathcal{A}_q \rightleftharpoons \beta_1 \mathcal{A}_1 + \ldots + \beta_q \mathcal{A}_q \quad \alpha_i, \beta_i \in \mathbb{Z}_+ ,$$

in a bounded box $\Omega \subset \mathbb{R}^N$ ($N \geq 1$). More precisely, we suppose that the concentrations $a_i = a_i(t,x) \geq 0$ of the species $\mathcal{A}_i$ at time $t \geq 0$ and point $x \in \Omega$ react according to the principle of mass action kinetics, i.e. satisfy the reaction-diffusion equations

$$\partial_t a_i - d_i \, \Delta_x a_i = (\beta_i - \alpha_i) \left(l \prod_{i=1}^{q} a_i^{\alpha_i} - k \prod_{i=1}^{q} a_i^{\beta_i} \right) \tag{1}$$

with the homogeneous Neumann boundary condition $\nabla_x \, a_i \cdot n = 0$ (on $\partial\Omega$, with n the outward normal to Ω). Here, d_i are constant diffusion rates, α_i, β_i the stoichiometric coefficients, and $k > 0$, $l > 0$ are strictly positive reaction rates.

Systems of the type (1) are well known in the numerous literature on reaction-diffusion systems. For the large time behaviour of global classical solutions (e.g. within invariant domains) we refer, for instance, to [Rothe, CHS] and the references therein. For global weak solutions see e.g. [MP, Pie, PS] with references. Many authors (e.g. [Zel, Mas, HMP, Web, HY88, HY94, KK] and the references therein) have deduced compactness and a-priori bounds from Lyapunov functionals. We recall in particular [Mor, FMS, FHM], where generalised Lyapunov structures of reaction-diffusion systems yield a-priori estimates to establish global existence of solutions. We mention also [Rio, Mul] and the references therein where peculiar Lyapunov functional are designed to show optimised stability and instability properties for reaction-diffusion systems.

In an approach which is related to the one proposed here (although it is non-constructive), the work of [Grö, GGH, GH] shows that systems quite more general than (1) have a unique asymptotically stable steady state.

The method we propose exploits the free energy functional of these systems. Going back to Boltzmann's work on gas dynamics, the basic idea of the entropy method consists in studying the large-time asymptotics of a dissipative PDE by looking for a nonnegative Lyapunov functional $H = H(f)$ and its nonnegative dissipation $D = D(f)$ (i.e. functionals which satisfy $\frac{d}{dt} H(f(t)) = -D(f(t))$ along the flow of the PDE), which are well-behaved in the following sense: first, $H(f) = 0 \iff f = f_\infty$ for some equilibrium f_∞ (usually, such a result is true only when all the conserved quantities have been taken into account), and secondly, there exists an entropy entropy-dissipation estimate of the form $D(f) \geq \Phi(H(f))$ for some nonnegative function Φ such that $\Phi(x) = 0 \iff x = 0$. If $\Phi'(0) \neq 0$, one usually gets exponential convergence toward f_∞ with a rate which can be explicitly estimated.

This method, which is an alternative to the linearisation around the equilibrium, has the advantage of being quite robust. This is due to the fact that it mainly relies on functional inequalities which have no direct link with the original PDE.

The entropy method has lately been used in many situations: nonlinear diffusion equations (such as fast diffusions [DelPD, CV], equations of fourth order [CCT], Landau equation [DV00], etc.), integral equations (such as the spatially homogeneous Boltzmann equation [TV1, TV2, V]), or kinetic equations ([CCG], [DV01, DV], [FNS]).

In a previous paper [DF], we have shown quantitative exponential convergence to equilibrium with explicit rates and constants for the systems modelling the reactions $2\,\mathcal{A} \rightleftharpoons \mathcal{B}$ and $\mathcal{A} + \mathcal{B} \rightleftharpoons \mathcal{C}$. For these systems, global L^∞-bounds on the concentrations are known. Moreover, we assumed the diffusion rates of all species to be strictly positive.

The topic of the present paper are extensions into two directions, both related to some degeneracy in the system of equations: First, we prove (slower) explicit exponential convergence when one diffusion rate degenerates to zero, but - due to the reaction - when the same constant steady state is conserved for the system.

As second extension, we investigate a system for four species $\mathcal{A} + \mathcal{B} \rightleftharpoons \mathcal{C} + \mathcal{D}$, for which a global L^∞-bound on the concentrations is - up to our knowledge - unknown, but where, at least in 1D, a polynomially growing L^∞-bound can be established. By a precise entropy-entropy dissipation estimate (where precise means that the constant of the estimate depends only logarithmically on the L^∞-bound on the solution), we show almost exponential decay towards the steady state in 1D. Further, that almost exponential decay interpolated with a polynomially growing H^1-bound yields in return an explicit, uniform in time L^∞-bound. Thus, we prove finally again explicit exponential decay towards the steady state. Note that slowly growing a-priori bounds have already been used in the context of entropy methods in kinetic theory (cf. [TV2]), as well as the case of interpolation between an explicit decay in weak norm and controlled growth in strong norm (cf. [DM]).

1.2 Degenerated Diffusion

In particular, we study degenerated diffusion rates for the chemical reaction $2\,\mathcal{A} \rightleftharpoons \mathcal{B}$ modelled by

$$\partial_t a - d_a \triangle_x a = -2(l\,a^2 - k\,b)\,, \tag{2}$$

$$\partial_t b - d_b \triangle_x b = l\,a^2 - k\,b\,, \tag{3}$$

$$n(x)\cdot\nabla_x a = 0, \qquad n(x)\cdot\nabla_x b = 0 \qquad x \in \partial\Omega\,, \tag{4}$$

$$a(0,x) = a_0(x) \geq 0, \qquad b(0,x) = b_0(x) \geq 0\,. \tag{5}$$

Here a, b denote the concentrations of the species $\mathcal{A}, \mathcal{B}$, which satisfy the homogeneous Neumann boundary conditions (4) and are subject to the nonnegative initial condition (5). The diffusion rates $d_a \geq 0$, $d_b \geq 0$ are non-negative constants.

We remark that the rescaling $t \to \frac{1}{k}t$, $x \to |\Omega|^{\frac{1}{N}}x$, $(a,b) \to \frac{k}{l}(a,b)$, allows - without loss of generality - to assume that

$$l = k = 1\,, \qquad |\Omega| = 1\,. \tag{6}$$

The flow of equations (2) – (5) conserves the total initial mass M

$$M = \int_\Omega (a(t,x) + 2b(t,x))\,dx = \int_\Omega (a_0(x) + 2b_0(x))\,dx\,, \tag{7}$$

which we assume strictly positive ($M > 0$) and which determines under the assumption that either $d_a > 0$ or $d_b > 0$ the unique equilibrium state (a_∞, b_∞) as the nonnegative constants satisfying $a_\infty + 2b_\infty = M$ and $a_\infty^2 = b_\infty$, i.e.

$$a_\infty = -\frac{1}{4} + \frac{1}{4}\sqrt{1+8M}\,, \qquad b_\infty = \frac{M - a_\infty}{2} = a_\infty^2\,. \tag{8}$$

Global existence of a unique classical solution to (2) – (6) is well known and a consequence of the L^∞-bound

$$0 \leq a(t) \leq L = \|a_0\|_\infty + 2\|b_0\|_\infty\,, \qquad 0 \leq b(t) \leq \frac{L}{2}\,, \qquad t \geq 0\,, \tag{9}$$

which can be shown, for instance, by the maximum principle applied to the individual equations (see e.g. [Kir]) or by comparison with the diffusionless system (see e.g. [BH]).

We consider as entropy functional the physical free energy of (2) – (6)

$$E(a,b) = \int_\Omega \left(a\,(\ln a - 1) + b\,(\ln b - 1)\right)\,dx\,. \tag{10}$$

Assuming strictly positive diffusion rates, we have shown in [DF, theorem1.1]:

Proposition 1.1 *Let Ω be a smoothly bounded and connected open set of $\mathbb{R}^N$ ($N \geq 1$), and $d_a > 0$, $d_b > 0$ be strictly positive diffusivity constants. Let the initial data a_0, b_0 be nonnegative functions of $L^\infty(\Omega)$ with strictly positive mass $\int_\Omega (a_o + 2b_0)\, dx = M > 0$ and denote $L = \|a_0\|_\infty + 2\|b_0\|_\infty$. Then, the unique nonnegative global solution $t \in \mathbb{R}_+ \mapsto (a(t), b(t))$ in $L^\infty(\Omega)$ to equations (2) – (6) obeys the following exponential decay toward equilibrium:*

$$\begin{aligned} &\frac{1}{2}\, \|a(t,\cdot) - a_\infty\|^2_{L^1(\Omega)} + \|b(t,\cdot) - b_\infty\|^2_{L^1(\Omega)} \\ &\leq \frac{(6+2\sqrt{2})M}{3+2\sqrt{2}}\, (E(a_0,b_0) - E(a_\infty,b_\infty))\, e^{-\frac{4t}{K_1(L,M)} \min\left\{1, \frac{d_a}{P(\Omega) K_2(M, d_a/d_b)}\right\}}, \end{aligned} \tag{11}$$

where $P(\Omega)$ is the Poincaré constant of Ω, and $K_1(L,M)$ is a constant defined as follows: we introduce the function $\Phi : (0,\infty)^2 \to \mathbb{R}$ defined by

$$\Phi(x,y) = \frac{x\,(\ln(x) - \ln(y)) - (x-y)}{(\sqrt{x} - \sqrt{y})^2}, \tag{12}$$

which is continuous on $(0,\infty)^2$ (with $\Phi(x,x) = 2$), strictly increasing in $x \in (0,\infty)$, and strictly decreasing in $y \in (0,\infty)$ [DF, lemma 2.1]. Then

$$K_1(L,M) = \max\left\{\frac{\Phi(L,a_\infty)}{a_\infty}, \Phi\left(\frac{L}{2}, b_\infty\right)\right\} \quad (= O(\ln(L)) \quad \text{for large } L)\ . \tag{13}$$

Moreover, $K_2(M, d_a/d_b) = K_2(\gamma)$ as defined in (46) with γ given in (47), i.e.

$$K_2(M, d_a/d_b) = \frac{d_a}{d_b}\frac{\sqrt{1+8M}}{2} + \sqrt{\frac{d_a^2}{d_b^2}\frac{1+8M}{4} + \frac{d_a}{d_b}\frac{\sqrt{1+8M}-1}{4}}\,. \tag{14}$$

In section 2, we extend proposition 1.1 to cover degenerate diffusion rates $d_a > 0, d_b = 0$ or $d_a = 0, d_b > 0$, for which the systems (2) – (6) still admits the steady state (8). In fact, we derive functional inequalities which quantify how the trend of the reaction transfers spatial diffusion of one specie to the other. We prove the theorems:

Theorem 1.1 ***(Degenerated Diffusion** $d_b \geq 0$)*
Let the assumptions of proposition 1.1 hold except that $d_b \geq 0$. Then,

$$\begin{aligned} &\frac{1}{2}\, \|a(t,\cdot) - a_\infty\|^2_{L^1(\Omega)} + \|b(t,\cdot) - b_\infty\|^2_{L^1(\Omega)} \\ &\leq \frac{(6+2\sqrt{2})M}{3+2\sqrt{2}}\, (E(a_0,b_0) - E(a_\infty,b_\infty))\, e^{-\frac{K_6(d_a,L,P)}{K_1(L,M)}\, t} \end{aligned} \tag{15}$$

where $K_1(L,M)$ is still defined by (13) and for all $\mu \in (0,1)$ and $\gamma \in (0,\infty)$:

$$K_6(d_a,L,P) = \begin{cases} \min\left\{4\mu, \frac{4\mu d_a}{K_2(\gamma)P}, \frac{(8L_1^2P)^2}{(8L_1^2P+d_a)^2}\frac{4(1-\mu)}{K_3(\gamma)}\right\} & \frac{d_a}{P} \le 8L_1^2\,, \\ \min\left\{4\mu, \frac{4(\frac{d_a}{P}-(1-\mu)8L_1^2}{K_2(\gamma)}, \frac{2(1-\mu)}{K_3(\gamma)}\right\} & \frac{d_a}{P} > 8L_1^2\,, \end{cases} \tag{16}$$

with the constants $K_2(\gamma)$ and $K_3(\gamma)$ defined in (46). Here, $\gamma \in (0,\infty)$ and $\mu \in (0,1)$ can be chosen in order to maximise K_6.

Theorem 1.2 ***(Degenerated Diffusion** $d_a \ge 0$)*
Let the assumptions of proposition 1.1 hold except that $d_a \ge 0$. Then,

$$\begin{aligned} &\frac{1}{2}\left\|a(t,\cdot)-a_\infty\right\|^2_{L^1(\Omega)} + \left\|b(t,\cdot)-b_\infty\right\|^2_{L^1(\Omega)} \\ &\le \frac{(6+2\sqrt{2})M}{3+2\sqrt{2}}\left(E(a_0,b_0)-E(a_\infty,b_\infty)\right)e^{-\frac{K_8(d_b/P,M)}{K_1(L,M)}t} \end{aligned} \tag{17}$$

where $K_1(L,M)$ is defined by (13) and for all $\mu \in (0,1)$, $\gamma \in (0,\infty)$ and $\nu \in (1,\infty)$:

$$K_8(d_b/P,M) = \min\left\{4\mu, 4(1-\mu)\frac{1-\eta_1}{K_2(\gamma)}\frac{1}{K_7(\nu)}, 4\mu\frac{d_b}{K_3(\gamma)P}\right\}, \tag{18}$$

where $\eta_1(d_b/P)$ is a constant given in (62). Here, $\mu \in (0,1)$, $\gamma \in (0,\infty)$ in $K_2(\gamma)$ and $K_3(\gamma)$ defined in (46), and $\nu \in (1,\infty)$ in $K_7(\nu)$ defined in (51) can be chosen in order to maximise K_8.

1.3 Slowly Growing A-Priori Bounds

We study another kind of degeneracy, namely the case when the L^∞-bound present in the entropy-entropy dissipation estimate is growing at most polynomially in time, so that the situation is comparable to that of [TV2]. In 1D, we still show exponential convergence with an explicit constant.

More precisely, we wish to investigate the chemical reaction $\mathcal{A}+\mathcal{B} \rightleftharpoons \mathcal{C}+\mathcal{D}$ (or equally denoted $\mathcal{A}_1+\mathcal{A}_2 \rightleftharpoons \mathcal{A}_3+\mathcal{A}_4$), which leads to the system

$$\partial_t a - d_a \triangle_x a = -l\,ab + k\,cd, \tag{19}$$
$$\partial_t b - d_b \triangle_x b = -l\,ab + k\,cd, \tag{20}$$
$$\partial_t c - d_c \triangle_x c = l\,ab - k\,cd, \tag{21}$$
$$\partial_t d - d_d \triangle_x d = l\,ab - k\,cd, \tag{22}$$

with a, b, c, d satisfying homogeneous Neumann conditions

$$n(x)\cdot\nabla_x a = 0, \quad n(x)\cdot\nabla_x b = 0, \quad n(x)\cdot\nabla_x c = 0, \quad n(x)\cdot\nabla_x d = 0 \quad x \in \partial\Omega, \tag{23}$$

and the nonnegative initial condition

$$\begin{aligned} a(0,x) &= a_0(x) \geq 0, \qquad b(0,x) = b_0(x) \geq 0, \\ c(0,x) &= c_0(x) \geq 0, \qquad d(0,x) = d_0(x) \geq 0\,. \end{aligned} \tag{24}$$

Again - without loss of generality - we assume

$$l = k = 1\,, \qquad |\Omega| = 1\,, \tag{25}$$

thanks to the rescaling $t \to \frac{1}{k}t$, $x \to |\Omega|^{\frac{1}{N}}x$, $(a,b,c,d) \to \frac{k}{l}(a,b,c,d)$.

The solutions of (19) – (24) conserve the quantities

$$M_{13} \;=\; \int_\Omega (a(t,x) + c(t,x))\,dx = \int_\Omega (a_0(x) + c_0(x))\,dx\,, \tag{26}$$

$$M_{14} \;=\; \int_\Omega (a(t,x) + d(t,x))\,dx = \int_\Omega (a_0(x) + d_0(x))\,dx\,, \tag{27}$$

$$M_{23} \;=\; \int_\Omega (b(t,x) + c(t,x))\,dx = \int_\Omega (b_0(x) + c_0(x))\,dx\,, \tag{28}$$

and as a consequence

$$M_{24} = M_{14} + M_{23} - M_{13} = \int_\Omega (b(t,x) + d(t,x))\,dx = \int_\Omega (b_0(x) + d_0(x))\,dx\,, \tag{29}$$

where the mass-indices ij correspond to the conserved quantity $\int_\Omega (a_i + a_j)\,dx$. Moreover, we shall assume strictly positive masses

$$M_{13} > 0\,, \qquad M_{14} > 0\,, \qquad M_{23} > 0\,, \qquad M_{24} > 0\,, \tag{30}$$

and introduce the total mass

$$M_{1234} = M_{13} + M_{24} = M_{14} + M_{23}\,, \tag{31}$$

and the following natural bounds $(\int_\Omega a_i\,dx \leq M_i)$

$$\begin{aligned} M_1 &= \min\{M_{13}, M_{14}\}\,, \qquad M_2 = \min\{M_{23}, M_{24}\}\,, \\ M_3 &= \min\{M_{13}, M_{23}\}\,, \qquad M_4 = \min\{M_{14}, M_{24}\}\,. \end{aligned} \tag{32}$$

Then, for (19) – (22) satisfying (26) – (28) with (30), there exists a unique equilibrium state $(a_\infty, b_\infty, c_\infty, d_\infty)$ defined by the unique positive constants solving $a_\infty + c_\infty = M_{13}$, $a_\infty + d_\infty = M_{14}$, $b_\infty + c_\infty = M_{23}$, and $a_\infty\, b_\infty = c_\infty\, d_\infty$:

$$\begin{aligned} a_\infty &= \tfrac{M_{13}M_{14}}{M_{1234}} > 0, \qquad & b_\infty &= M_{23} - \tfrac{M_{13}M_{23}}{M_{1234}} = \tfrac{M_{23}M_{24}}{M_{1234}} > 0\,, \\ c_\infty &= \tfrac{M_{13}M_{23}}{M_{1234}} > 0, \qquad & d_\infty &= M_{14} - \tfrac{M_{13}M_{14}}{M_{1234}} = \tfrac{M_{14}M_{24}}{M_{1234}} > 0\,. \end{aligned} \tag{33}$$

Finally, the entropy (free energy) functional associated to (19) – (25) is

$$E(a_1, a_2, a_3, a_4) = \int_\Omega \sum_1^4 a_i(\ln(a_i) - 1)\, dx\,. \tag{34}$$

Global existence of unique classical solutions of (19)–(25) has been shown in 1D in the work of [Ama] and [Mor, theorem 2.4] . Global existence of weak solutions in any dimension follows e.g. from [Pie]. Within the presented results, global existence of classical solutions in 1D is a consequence of lemma 3.5 in section 3.1.

In section 3.1, we first study a-priori bounds entailed by the decay of the entropy functional (34). Then in 1D, these bounds allow to bootstrap an explicit, polynomially in time growing L^∞-bound on the concentrations a_i (lemma 3.5). In section 3.2, we establish in several lemmata an entropy entropy-dissipation estimate with a constant depending logarithmically on the L^∞-norm of the solution and prove :

Proposition 1.2 *Let Ω be the interval $[0,1]$, and d_a, d_b, d_c, d_d be strictly positive diffusion rates. Let the initial data a_0, b_0, c_0, and d_0 be nonnegative functions of $L^\infty(\Omega)$ with strictly positive masses M_{13}, M_{14}, M_{23}, and M_{24}. Then, the unique nonnegative global solution with (at most) polynomially increasing $L^\infty(\Omega)$-norm $\|a_i(t)\|_{L^\infty(\Omega)}$ (see lemma 3.5) to the equations (19) – (25) satisfies the following almost exponential decay towards the steady state $a_{i,\infty}$ given in (33) :*

$$\begin{aligned} &\frac{\|a(t,\cdot)-a_\infty\|^2_{L^1(\Omega)}}{M_1} + \frac{\|b(t,\cdot)-b_\infty\|^2_{L^1(\Omega)}}{M_2} + \frac{\|c(t,\cdot)-c_\infty\|^2_{L^1(\Omega)}}{M_3} + \frac{\|d(t,\cdot)-d_\infty\|^2_{L^1(\Omega)}}{M_4} \qquad (35) \\ &\leq 2\sqrt{2}(E(a_{i,0}) - E(a_{i,\infty}))\, \exp\left(-\frac{t}{\ln(e+t)}\, \frac{4\min\left\{\frac{1}{K_{10}}, \frac{\min\{d_a,d_b,d_c,d_d\}}{K_{11}\,P(\Omega)}\right\}}{(\sqrt{2}+1)^2\left(\max\limits_{i=1,2,3,4}\left\{\frac{K_{19,i}}{a_{i,\infty}}, \ln 2\right\} + \frac{21}{2}\right)} \right), \end{aligned}$$

where K_{10} is defined in (112) (depending on K_9 given in (91)), K_{11} is defined in (113), $K_{19,i}$ is defined in lemma 3.5 (depending on $K_{17,i}$ (66), K_{18} (73), $K_{21,i}$ (88), and K_{20} (86)), and $P(\Omega)$ denotes the Poincaré constant of Ω.

Remark 1.1 *The various constants are not optimised, but we write them in order to convince the reader that they are explicitly computable.*

Furthermore, the almost exponential decay of theorem 1.2 interpolates with an explicit, polynomially growing H^1-bound and we obtain an explicit, uniform in time L^∞-bound, and thus, in return, exponential decay towards the steady state :

Theorem 1.3 *Under the assumptions of proposition 1.2, the concentrations* a_i*,* $i = 1,2,3,4$ *are globally bounded in* L^∞ *with*

$$\|a_i(t)\|_{L^\infty(\Omega)} \le \begin{cases} \frac{21}{19} K_{19,i}\,, & 0 < t \le (2/19)^{2/21}\,, \\ K_{13,i}\,, & t > (2/19)^{2/21}\,, \end{cases} \tag{36}$$

$$K_{13,i} = a_{i,\infty} + G(\Omega)\, K_{22}^{\frac14}\, K_{19,i}^{\frac14} \left(2^{\frac32}\, M_i(E(a_{i,0}) - E(a_{i,\infty}))\right)^{\frac18} 3^{\frac14}\, K_{14}\,, \tag{37}$$

$$K_{14} = \sup_{t\in[0,\infty)} \left\{ \left(1 + t^{\frac{59}{2}}\right)^{\frac14} \exp\left(-\frac{t}{\ln(e+t)}\, \frac{\min\left\{\frac{1}{K_{10}}, \frac{\min\{d_a,d_b,d_c,d_d\}}{K_{11}P(\Omega)}\right\}}{2(\sqrt{2}+1)^2 \left(\max_{i=1,2,3,4} \left\{\frac{K_{19,i}}{a_{i,\infty}}\right\} + \frac{21}{2}\right)} \right) \right\},$$

where $K_{19,i}$ *is defined in lemma 3.5 (depending on* $K_{15,i}$ *given in (63),* $K_{17,i}$ *in (66),* K_{18} *in (73),* $K_{21,i}$ *in (88), and* K_{20} *in (86)),* K_{22} *is defined in (143),* M_i *is defined in (32),* K_{10} *is defined in (112) (depending on* K_9 *given in (91)),* K_{11} *is defined in (113),* Φ *is given in (12), and* $P(\Omega)$ *denotes the Poincaré constant of* Ω *and* $G(\Omega)$ *denotes the Gagliardo-Nirenberg-Moser constant in (144). Moreover, the concentrations* a_i *decay exponentially towards the steady state* $a_{i,\infty}$ *given in (33) :*

$$\frac{\|a(t,\cdot)-a_\infty\|^2_{L^1(\Omega)}}{M_1} + \frac{\|b(t,\cdot)-b_\infty\|^2_{L^1(\Omega)}}{M_2} + \frac{\|c(t,\cdot)-c_\infty\|^2_{L^1(\Omega)}}{M_3} + \frac{\|d(t,\cdot)-d_\infty\|^2_{L^1(\Omega)}}{M_4} \tag{38}$$
$$\le 2\sqrt{2}(E(a_{i,0}) - E(a_{i,\infty}))\, \exp\left(-t \left(\frac{4\min\left\{\frac{1}{K_{10}}, \frac{\min\{d_a,d_b,d_c,d_d\}}{K_{11}P(\Omega)}\right\}}{\max_{i=1,2,3,4}\{\Phi(K_{13,i}, a_{i,\infty})\}} \right)\right).$$

Remark 1.2 *Note that exponential decay towards equilibrium in all Sobolev norms follows subsequently by interpolation of the decay of theorem 1.3 with polynomially growing* H^k*-bounds, which follow iteratively for* $k > 1$ *from (36) and (142) inserted into the Fourier-representation as used in lemma 3.3.*

Notations: The letters K, K_1, $K_2, \ldots$ denote various positive constants. We shall equally write $a_1 = a,\ a_2 = b,\ a_3 = c,\ a_4 = d$, for the concentrations, whereever the index notation is more convenient. It will also be convenient to introduce capital letters as a short notation for square roots of lower case concentrations and overlines for spatial averaging (remember that $|\Omega| = 1$)

$$A_i = \sqrt{a_i}\,, \qquad A_{i,\infty} = \sqrt{a_{i,\infty}}\,, \qquad \overline{A_i} = \int_\Omega A_i\, dx\,, \qquad i = 1,2,3,4\,.$$

Finally, we denote $\|f\|_2^2 = \int_\Omega f^2\, dx$ for a given function $f : \Omega \to \mathbb{R}$.

Outline: In section 2, we prove theorems 1.1 and 1.2. Next, in section 3, we prove proposition 1.2 and theorem 1.3.

2 The degenerated diffusion case $2\mathcal{A} \leftrightarrow \mathcal{B}$

At first we shall **sketch the proof of proposition 1.1**. We recall that

$$\frac{d}{dt}(E(a(t),b(t)) - E(a_\infty, b_\infty)) = -\,D(a(t),b(t))\,,$$

with $D(a,b)$ denoting the entropy dissipation

$$D(a,b) = d_a \int_\Omega \frac{|\nabla_x a|^2}{a}\,dx + d_b \int_\Omega \frac{|\nabla_x b|^2}{b}\,dx + \int_\Omega (a^2 - b) \ln \frac{a^2}{b}\,dx\,, \qquad (39)$$

and use crucially the following entropy entropy-dissipation lemma :

Lemma 2.1 *For all (measurable) functions $a,b : \Omega \to \mathbb{R}$, which satisfy $0 \le a \le L$, $0 \le b \le \frac{L}{2}$, and $\int_\Omega (a+2b) = M$,*

$$D(a,b) \ge \frac{4}{K_1} \min\left\{1, \frac{d_a}{P(\Omega)K_2}\right\}(E(a,b) - E(a_\infty, b_\infty)), \qquad (40)$$

where $P(\Omega)$ is the Poincaré constant of Ω, a_∞, b_∞ are given by (8), and the explicit constants $K_1(L,M)$, $K_2(M, d_a/d_b)$ are defined by the formulas (13) and (14).

Thus, lemma 2.1 implies exponential convergence of the relative entropy, i.e.

$$E(a(t),b(t)) - E(a_\infty,b_\infty) \le (E(a_0,b_0) - E(a_\infty,b_\infty))\, e^{-\frac{4t}{K_1}\min\{1,\frac{d_a}{P(\Omega)K_2}\}}\,. \qquad (41)$$

The statement of proposition 1.1 is finally a consequence of estimate (41) and the following analog of the Cziszar-Kullback-Pinsker inequality ([Csi] and [Kul]) in information theory:

Lemma 2.2 *For all (measurable) functions $a,b : \Omega \to \mathbb{R}$ such that $0 \le a$, $0 \le b$ and $\int_\Omega (a+2b) = M$,*

$$E(a,b) - E(a_\infty,b_\infty) \ge \frac{3+2\sqrt{2}}{(6+2\sqrt{2})M}\left(\frac{1}{2}\,\|a-a_\infty\|_1^2 + \|b-b_\infty\|_1^2\right), \qquad (42)$$

where a_∞ and b_∞ are defined by (8).

The proof of the central lemma 2.1 divides into the following steps: Firstly, we estimate below the entropy dissipation $D(a,b)$ using the identity $|\nabla_x a_i|^2/a_i = 4\,|\nabla_x A_i|^2$ (with $A_i = \sqrt{a_i}$, $i=1,2$) and Poincaré's inequality as well as the inequality $(a-b)(\ln(a)-\ln(b)) \geq 4(A-B)^2$. We obtain in this way the estimate:

$$D(a,b) \geq 4\left\|A^2 - B\right\|_2^2 + \frac{4d_a}{P(\Omega)}\left\|A-\overline{A}\right\|_2^2 + \frac{4d_b}{P(\Omega)}\left\|B-\overline{B}\right\|_2^2 . \tag{43}$$

On the other hand, the continuity of the function Φ defined in (12) (see [DF, lemma 2.1]) and (9) imply an upper bound for the relative entropy

$$E(a,b) - E(a_\infty, b_\infty) \leq K_1(L,M)\left(A_\infty^2\,\|A-A_\infty\|_2^2 + \|B-B_\infty\|_2^2\right), \tag{44}$$

with $K_1(L,M)$ given in (13). Moreover, under the crucial use of the conservation of mass, equation (44) can be estimated further to

$$\frac{1}{K_1}\left(E(a,b) - E(a_\infty,b_\infty)\right) \leq \|A^2-B\|_2^2 + K_2\|A-\overline{A}\|_2^2 + K_3\|B-\overline{B}\|_2^2 , \tag{45}$$

with the constants (for some $\gamma > 0$ to be chosen)

$$K_2(\gamma) = A_\infty \gamma, \qquad K_3(\gamma) = 4B_\infty + 1 + \frac{A_\infty}{\gamma} . \tag{46}$$

Finally, by comparing and balancing (45) with (43), we determine γ in order to set the fraction $K_2/K_3 = d_a/d_b$ according to (43), i.e. we set

$$\gamma = \frac{d_a}{d_b}\left(2A_\infty + \frac{1}{2A_\infty}\right) + \sqrt{\frac{d_a^2}{d_b^2}\left(2A_\infty + \frac{1}{2A_\infty}\right)^2 + \frac{d_a}{d_b}} , \tag{47}$$

and the constant (14) is chosen according to (46) and (8). □

In the cases of diffusivity rates $d_a = 0$ or $d_b = 0$, the proof of proposition 1.1 lacks one of the terms $d_a\|A-\overline{A}\|_2^2$ or $d_b\|B-\overline{B}\|_2^2$ in (43) although both terms would be required to match with (45). The following functional inequalities "produce" spatial diffusion in terms of $\|A_i - \overline{A_i}\|_2^2$, $i=1,2$ at the costs of the reaction dissipation term $\|A^2-B\|_2^2$ and the spatial diffusion $\|A_j - \overline{A_j}\|_2^2$, $j \neq i$ of the other species. We first consider the case $d_b = 0$ (lemma 2.3) and then the case $d_a = 0$ (lemma 2.4).

Lemma 2.3 *Let A, B denote the square roots of the solution of (2) – (6) satisfying the global bound (9). Then,*

$$K_4\|A^2-B\|_2^2 + K_5\|A-\overline{A}\|_2^2 \geq \|B-\overline{B}\|_2^2 \tag{48}$$

for all $\theta \in [0,1)$ and with the constants

$$K_4(\theta) = \frac{2}{1-\theta^2}\,, \qquad K_5(L,\theta) = \frac{16L^2}{(1+\theta)^2}\,. \tag{49}$$

Proof: For $\theta \in [0,1)$, we expand $\|A^2 - B\|_2^2$ as

$$\begin{aligned}\|A^2 - \overline{B} + \overline{B} - B\|_2^2 &= \|A^2 - \overline{B}\|_2^2 + 2\theta \int_\Omega (A^2 - \overline{B})(\overline{B} - B)\,dx \\ &\quad +2(1-\theta)\int_\Omega (A^2 - \overline{A}^2)(\overline{B} - B)\,dx + \|\overline{B} - B\|_2^2\,,\end{aligned}$$

where we have used that $\int_\Omega \overline{B}(\overline{B} - B)\,dx = \int_\Omega \overline{A}^2(\overline{B} - B)\,dx = 0$. Estimating both integrals with Young's inequality (i.e. $2xy \geq -\gamma^{-1}x^2 - \gamma y^2$ with $\gamma = \theta$ for the first integral, and $\gamma = (1+\theta)/2$ for the second), and comparing the coefficients with (48), the following inequalities have to be satisfied:

$$K_5\|\overline{A} - A\|_2^2 - \frac{2K_4(1-\theta)}{1+\theta}\|A^2 - \overline{A}^2\|_2^2 \geq 0\,, \qquad K_4(-\theta^2 - (1-\theta)\frac{1+\theta}{2} + 1) \geq 1\,.$$

Using the global bound (9) to estimate $\|A^2 - \overline{A}^2\|_2^2 \leq 4L^2\|\overline{A} - A\|_2^2$, we are led to use the constants (49). □

Proof of theorem 1.2: Let assume firstly that $\frac{d_a}{P} \leq 8L^2$, which means that the fraction K_4/K_5 of the constants (49) matches (for $\theta = \frac{8L^2P - d_a}{8L^2P + d_a}$ and therefore $K_4 = \frac{(8L^2P + d_a)^2}{(8L^2P)^2}$) with the coefficients of

$$D \geq 4\|A^2 - B\|_2^2 + 4\frac{d_a}{P}\|A - \overline{A}\|_2^2\,.$$

Then, we may estimate (for all $\mu \in (0,1)$)

$$\begin{aligned} D &\geq 4\mu\|A^2 - B\|_2^2 + 4\mu\frac{d_a}{P}\|A - \overline{A}\|_2^2 + 4(1-\mu)\frac{(8L^2P)^2}{(8L^2P + d_a)^2}\|B - \overline{B}\|_2^2 \\ &\geq \frac{K_6}{K_1}(E - E_\infty)\,.\end{aligned}$$

In the second case $\frac{d_a}{P} \geq 8L^2$, we use instead the inequalities:

$$\begin{aligned} D &\geq 4\mu\|A^2 - B\|_2^2 + 4(\frac{d_a}{P} - (1-\mu)8L^2)\|A - \overline{A}\|_2^2 \\ &\quad +4(1-\mu)\|A^2 - B\|_2^2 + 4(1-\mu)8L^2\|A - \overline{A}\|_2^2\,,\end{aligned}$$

and the second line is bounded below by $2(1-\mu)\|B - \overline{B}\|_2^2$ via lemma 2.3 (with $\theta = 0$), which leads to the constant (16) in this case. ■

We now consider the case $d_a = 0$:

Lemma 2.4 *Let A, B denote the square roots of the solution of (2) – (6) satisfying the conservation law (7). Then, for all $\eta < 1$,*

$$\frac{1}{1-\eta}K_7\|A^2 - B\|_2^2 + \frac{\eta+1}{\eta}K_7\|B - \overline{B}\|_2^2 \geq \|A - \overline{A}\|_2^2\,, \tag{50}$$

with the constant $K_7(M)$ defined by (here, one can take any $\nu > 1$ in order to optimise this constant):

$$K_7(M) = \max\left\{\frac{\nu}{\nu-1}, \frac{2}{M}\frac{\nu}{\nu-1}, 2\nu^2 M + \nu - \frac{1}{2}, \frac{2}{M}\left(4\nu^2 + \nu\right) - \frac{1}{2}\right\}. \tag{51}$$

Proof: The proof deals with three cases: 1) $\overline{B}$ is "big" and 2a/b) $\overline{B}$ is "small" with $\overline{A^2}$ is "small" respectively "big".

1) For $\overline{B}$ "big", we apply the ansatz (pointwise for all $x \in \Omega$)

$$A^2(x) = \overline{B}(1 + \delta(x))\,, \qquad \delta(x) \in [-1, \infty)\,, \quad \forall x \in \Omega\,. \tag{52}$$

Then, after expanding $\|A^2 - B\|_2^2$ as

$$\|\overline{B} - B + \overline{B}\delta\|_2^2 = \|B - \overline{B}\|_2^2 + 2\overline{B}\int_\Omega \delta(\overline{B} - B)\,dx + \overline{B}^2\overline{\delta^2}\,,$$

and using Young's inequality for the above integral:

$$2\overline{B}\int_\Omega \delta(\overline{B} - B)\,dx \geq -\left(1 + \frac{1-\eta^2}{\eta}\right)\|B - \overline{B}\|_2^2 - \overline{B}^2\frac{\eta}{1+\eta-\eta^2}\overline{\delta^2}\,,$$

we get for the left hand side of (50) the following estimate

$$\begin{aligned}\tfrac{1}{1-\eta}K_7\left(\|B - \overline{B}\|_2^2 + 2\overline{B}\textstyle\int_\Omega \delta(\overline{B} - B)\,dx + \overline{B}^2\overline{\delta^2}\right) + \tfrac{\eta+1}{\eta}K_7\|B - \overline{B}\|_2^2 \\ \geq K_7\tfrac{1+\eta}{1+\eta-\eta^2}\overline{B}^2\overline{\delta^2} \geq K_7\overline{B}^2\overline{\delta^2}\,,\end{aligned} \tag{53}$$

since $(1+\eta)/(1+\eta-\eta^2) > 1$. Next, on the right hand side of (50), we Taylor-expand $A = \sqrt{\overline{B}}\sqrt{1+\delta}$ for all $x \in \Omega$ according to

$$\sqrt{1+\delta} = 1 + \frac{\delta}{2} - \frac{1}{4}\frac{1}{\sqrt{1+\zeta}^3}\frac{\delta^2}{2} \qquad \zeta(x) \in (0, \delta(x))\,, \tag{54}$$

and observe that the remainder term $R = R(\delta)$ defined by

$$R(\delta) = \frac{1}{\sqrt{1+\zeta}^3} = \frac{8}{\delta^2}\left(1 + \frac{\delta}{2} - \sqrt{1+\delta}\right)\,, \qquad \delta \in [-1, \infty) \tag{55}$$

is monotone decreasing on $\delta \in [-1, \infty)$ with $R(-1) = 4$ and $R(\infty) = 0$. We obtain therefore for the left hand side of (50) that

$$\begin{aligned}
\overline{A^2} - \overline{A}^2 &= \overline{B}(1+\overline{\delta}) - \overline{B}\left(1 + \overline{\delta} + \frac{\overline{\delta}^2}{4} - \frac{1}{4}\overline{\delta^2 R} - \frac{1}{8}\overline{\delta}\,\overline{\delta^2 R} + \frac{1}{64}\overline{\delta^2 R}^2\right) \\
&\leq \frac{\overline{B}}{4}\overline{\delta^2 R} + \frac{\overline{B}}{8}\overline{\delta}\,\overline{\delta^2 R} \;\leq\; \overline{B}\,\overline{\delta^2} + \frac{\overline{B}\,\overline{\delta}}{2}\overline{\delta^2} \leq \overline{\delta^2}\left(\overline{B} + \frac{\overline{B}\,\overline{\delta}}{2}\right) \\
&\leq \overline{\delta^2}\frac{M - \overline{B}^2 + \overline{B}}{2}, \qquad (56)
\end{aligned}$$

since $R \leq 4$ and $\overline{B}\,\overline{\delta} = \overline{A^2} - \overline{B} = M - \overline{B^2} - \overline{B} \leq M - \overline{B}^2 - \overline{B}$ by Jensen's inequality. Finally, by (53) and (56), equation (50) holds for

$$K_7 \geq \frac{1}{2}\frac{M - \overline{B}^2 + \overline{B}}{\overline{B}^2}, \qquad (57)$$

so that we have to find a different estimate if $\overline{B}$ is small.

2) As a preliminary step, we see that

$$\begin{aligned}
\|A^2 - B\|_2^2 &= \|A^2 - \overline{B}\|_2^2 + 2\int_\Omega (A^2 - \overline{B})(\overline{B} - B)\,dx + \|B - \overline{B}\|_2^2 \\
&\geq (1-\eta)\|A^2 - \overline{B}\|_2^2 - \left(\frac{1}{\eta} - 1\right)\|B - \overline{B}\|_2^2
\end{aligned}$$

by Young's inequality, and, hence, that it is sufficient to show that

$$K_7\|A^2 - \overline{B}\|_2^2 + K_7\|B - \overline{B}\|_2^2 \geq \|A - \overline{A}\|_2^2. \qquad (58)$$

Expanding and estimating further by Jensen's inequality ($\overline{A^4} \geq \overline{A^2}^2$), it is sufficient that

$$K_7 \geq \frac{\overline{A^2}}{\overline{A^2}^2 - 2\overline{B}\,\overline{A^2} + \overline{B^2}}. \qquad (59)$$

2a) Let us firstly assume that $\overline{A^2} \leq \frac{M}{2}$. Then, in (59), we may neglect the quadratic term $\overline{A^2}^2 \geq 0$ and use the conservation law $\overline{B^2} = M - \overline{A^2}$ to estimate (assuming $\overline{B} \leq \frac{1}{2\nu}$)

$$\frac{\overline{A^2}}{-2\overline{B}\,\overline{A^2} + M - \overline{A^2}} \leq \frac{1}{-\frac{1}{\nu} + \frac{M}{\overline{A^2}} - 1} \leq \frac{1}{-\frac{1}{\nu} + 1} \leq \frac{\nu}{\nu - 1} \qquad \text{for} \quad \overline{B} \leq \frac{1}{2\nu}. \qquad (60)$$

2b) In the case when $\overline{A^2} \geq \frac{M}{2}$, we neglect $\overline{B^2} \geq 0$ and estimate (assuming $2\overline{B} \leq \frac{M}{2\nu} \leq \frac{\overline{A^2}}{\nu}$)

$$\frac{1}{\overline{A^2} - 2\overline{B}} \leq \frac{1}{\overline{A^2}\left(1 - \frac{1}{\nu}\right)} \leq \frac{2}{M}\frac{\nu}{\nu - 1} \qquad \text{for} \quad \overline{B} \leq \frac{M}{4\nu}. \tag{61}$$

Combining the cases, we obtain from (60) and (61) the first two contributions for the constant K_7 (51), which covers the case $\overline{B} \leq \min\{\frac{1}{2\nu}, \frac{M}{4\nu}\}$. In the other case, inserting $\overline{B} \geq \min\{\frac{1}{2\nu}, \frac{M}{4\nu}\}$ into (57) leads to the last two terms in (51). □

Proof of theorem 1.2: By lemma 2.4, we have

$$\|A^2 - B\|_2^2 + \frac{1-\eta^2}{\eta}\|B - \overline{B}\|_2^2 \geq \frac{1-\eta}{K_7}\|A - \overline{A}\|_2^2 .$$

Then, if we chose η_1 such that $\frac{d_b}{P} = \frac{1-\eta_1^2}{\eta_1}$, i.e.

$$\eta_1 = -\frac{d_b}{2P} + \sqrt{\frac{d_b^2}{4P^2} + 1}\,, \tag{62}$$

we obtain, for any $\mu \in (0,1)$,

$$\begin{aligned} D &\geq 4\mu\|A^2 - B\|_2^2 + 4(1-\mu)\frac{1-\eta_1}{K_7}\|A - \overline{A}\|_2^2 + 4\mu\frac{d_b}{P}\|B - \overline{B}\|_2^2 \\ &\geq K_8\left(\|A^2 - B\|_2^2 + K_2\|A - \overline{A}\|_2^2 + K_3\|B - \overline{B}\|_2^2\right), \end{aligned}$$

and estimate (44) completes the proof. ■

3 The case $\mathcal{A} + \mathcal{B} \leftrightarrow \mathcal{C} + \mathcal{D}$

3.1 A-priori estimates

The a-priori-estimates of lemma 3.1 below are direct consequences of the entropy decay. The statements are more easily readable when using the notation a_i, $i = 1,2,3,4$ instead of a, b, c, and d, (and A_i, $a_{i,0}$, d_i, or even $K_{n,i}$ for constants).

Lemma 3.1 ***(A-priori estimates from the entropy decay)***
Let a_i, $i = 1,2,3,4$ be solutions of the system (19)-(25) with initial data

$a_{i,0}\ln(a_{i,0}) \in L^1(\Omega)$. *Then, with* M_i *defined in (32) and for all* $T > 0$,

$$\begin{aligned} \|\nabla_x A_i\|^2_{L^2([0,T]\times\Omega)} &\le \frac{\int_\Omega \sum_{i=1}^4 a_{i,0}\ln(a_{i,0})\,dx + 4e^{-1}}{4d_i} := K_{15,i}\,, && (63)\\ \sup_{t\in[0,T]} \|a_i\ln(a_i)\|_{L^1(\Omega)} &\le \int_\Omega \sum_{i=1}^4 a_{i,0}\ln(a_{i,0})\,dx + 5e^{-1} := K_{16}\,, && (64)\\ \sup_{t\in[0,T]} \|A_i\|^2_{L^2(\Omega)} &\le M_i\,. && (65) \end{aligned}$$

Proof: Calculating the time-derivative of the entropy-functional (34) yields

$$\int_\Omega \sum_{i=1}^4 a_i\ln(a_i)(T)\,dx + 4\sum_{i=1}^4 d_i \int_0^T \int_\Omega |\nabla_x A_i|^2\,dxdt \le \int_\Omega \sum_{i=1}^4 a_{i,0}\ln(a_{i,0})\,dx\,,$$

which ensures directly that estimate (65) holds, once the conservation of masses (26)-(29) has been taken into account. Moreover, estimates (63) and (64) follow from $-a_i|\ln(a_i)| \le e^{-1}$. $\Box$

The a-priori estimates of lemma 3.1 combine and yield

Lemma 3.2 *Let* $N = 1$, *i.e.* Ω *be the interval* $[0,1]$. *Then, for* $i = 1,2,3,4$ *the solutions* a_i *of (19)-(25) with initial data* $a_{i,0}\ln(a_{i,0}) \in L^1(\Omega)$ *satisfy for* $T > 0$,

$$\|a_i\|^2_{L^2([0,T]\times[0,1])} \le K_{17,i}(1+T)\,, \qquad K_{17,i} = 2\,M_i(M_i + K_{15,i})\,, \qquad (66)$$

with M_i *defined in (32) and* $K_{15,i}$ *in (63).*

Moreover, for all $\varepsilon > 0$,

$$\|a_i\|^{3-\varepsilon}_{L^{3-\varepsilon}([0,T]\times[0,1])} \le K(\varepsilon)(1+T)$$

for some constant $K(\varepsilon) > 0$ *that we do not make explicit.*

Proof: By $A_i(t,x) - A_i(t,y) = \int_y^x \partial_u A_i(t,u)\,du$, we consider firstly

$$|A_i(t,x) - \int_0^1 A_i(t,y)\,dy| = \left|\int_0^1 \int_{u=y}^x \partial_u A_i(t,u)\,dudy\right| \le \int_{u=0}^1 |\partial_u A_i(t,u)|\,du,$$

and obtain

$$\begin{aligned} |A_i(t,x)| &\le \int_0^1 |\partial_u A_i(t,u)|\,du + \int_0^1 |A_i(t,u)|\,du\,,\\ |A_i(t,x)|^2 &\le 2\int_0^1 |\partial_u A_i(t,u)|^2\,du + 2\int_0^1 |A_i(t,u)|^2\,du\,. && (67) \end{aligned}$$

Combining the L^∞-bound (67) with (65) and (63) of lemma 3.1, we get

$$\int_0^T \int_0^1 |a_i(t,x)|^2\, dxdt \le \int_0^T \sup_{y\in[0,1]} |A_i(t,y)|^2 \left(\int_0^1 |A_i(t,x)|^2\, dx\, dt \right) \tag{68}$$
$$\le 2M_i \int_0^T \int_0^1 |\partial_u A_i(t,u)|^2\, dudt + 2M_i^2 T \le K_{17}(1+T)\,.$$

Then, we obtain an improved version of (63)

$$\int_0^T \int_0^1 |\partial_u A_i^3|\, dudt = \int_0^T \int_0^1 3|A_i|^2 |\partial_u A_i|\, dudt$$
$$\le 3 \left(\int_0^T \int_0^1 |A_i|^4\, dudt \right)^{\frac{1}{2}} \left(\int_0^T \int_0^1 |\partial_u A_i|^2\, dudt \right)^{\frac{1}{2}} \le K(1+T)\,, \tag{69}$$

by Hölders inequality and (68). In a next step, similar to (67) and (68),

$$\int_0^T \int_0^1 |a_i(t,x)|^{1+\frac{3}{2}}\, dxdt \le \int_0^T \sup_{y\in[0,1]} |A_i^3(t,y)| \int_0^1 |A_i(t,x)|^2\, dx\, dt$$
$$\le 2M_i \int_0^T \int_0^1 \left(|A_i^3| + |\partial_u A_i^3(t,u)|\right) dudt \le K(1+T)\,, \tag{70}$$

which again allows to increase the exponent 3 in (69). The statement of the lemma is finally obtained by bootstrapping (69) and (70). $\square$

With the next technical lemma, we provide an explicit, polynomially growing bound for L^r-norms ($r \ge 1$) of the solution of the 1D heat equation.

Lemma 3.3 ***(Explicit bounds for 1D heat equation)***
Let a denote the solution of the 1D heat equation ($t > 0, x \in [0,1]$, with constant diffusivity d_a) with homogeneous Neumann boundary condition, i.e.

$$\begin{aligned} a_t - d_a a_{xx} &= g\,, \\ a_x(t,0) = a_x(t,1) &= 0\,, \\ a(0,x) &= a_0(x)\,, \end{aligned} \tag{71}$$

and assume for the initial data and for the source term g that

$$a_0(x) \in L^1 \cap L^\infty([0,\infty) \times [0,1])\,, \qquad g(t,x) \in L^1 \cap L^p([0,\infty)\times[0,1])\,.$$

Then, for the exponents $r, p \ge 1$ and $q \in [1,3)$ satisfying $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$ and for all $T > 0$, the $L^r([0,T]\times[0,1])$-norm of the solution grows at most polynomially in T like

$$\begin{aligned} \|a(t,x)\|_{L^r([0,T]\times[0,1])} \;\le\; & T^{1/r} \left(\|a_0\|_{L^\infty[0,1]} + \|a_0\|_{L^1[0,1]} + \|g\|_{L^1([0,T]\times[0,1])} \right) \\ & + K_{18}(q)\,(1 + T^{\frac{1}{q}+\frac{1}{2}}) \|g\|_{L^p([0,T]\times[0,1])}\,, \end{aligned} \tag{72}$$

with the constant $K_{18}(q)$ given by

$$K_{18}(q) = 2^{\frac{4q^2-3q-2}{q(2q-1)}} \left(\frac{\pi}{q}\right)^{\frac{2+q}{2q(2q-1)}} \left(\frac{1}{3-q}\right)^{\frac{2+q}{q(2q-1)}} (2+q)^{\frac{3-q}{q(2q-1)}} + 2^{3+\frac{2}{q}} \left(\frac{1}{2+q}\right)^{\frac{1}{q}} \quad (73)$$

Note that $K_{18}(q) > 11$ for $q \in [1,3)$.

Proof: The proof uses Fourier series, which simplify when (71) is mirrored evenly around $x = 0$, i.e. when the functions are extended like

$$\tilde{a}(t,x) = \begin{cases} a(t,x) & x \in [0,1] \\ a(t,-x) & x \in [-1,0] \end{cases} \quad (74)$$

and when $\tilde{g}$ and $\tilde{a}_0$ are defined analogously. Then, the eigenvalue-problem $\tilde{\varphi}_{xx} = \lambda \tilde{\varphi}$ on $[-1,1]$ with homogeneous Neumann boundary and periodicity conditions is satisfied by the eigenvalue-eigenfunction-pairs

$$(\lambda_k, \tilde{\varphi}_k(x)) = (-(k\pi)^2, \cos(k\pi x)) \qquad \text{for} \quad k = 0,1,2,\ldots$$

and yields the Fourier representation

$$\begin{aligned} \tilde{a}(t,x) &= \sum_{k=0}^{\infty} e^{\lambda_k t} \left(\int_{-1}^{1} \tilde{a}_0(y) \tilde{\varphi}_k(y)\, dy \right) \tilde{\varphi}_k(x) \\ &\quad + \sum_{k=0}^{\infty} \int_0^t e^{\lambda_k (t-s)} \left(\int_{-1}^{1} \tilde{g}(s,y) \tilde{\varphi}_k(y) dy \right) ds\, \tilde{\varphi}_k(x)\,. \end{aligned} \quad (75)$$

By construction, the solution $a(t,x)$ is the restriction of $\tilde{a}(t,x)$ to $x \in [0,1]$. Considering in (75) that $\tilde{a}_0, \tilde{g}$ and $\tilde{\varphi}_k$ are even functions, we obtain with $\cos(x)\cos(y) = \cos(x-y) - \sin(x)\sin(y)$ that

$$\tilde{a}(t,x) = 2\int_0^1 a_0(y)\, dy + \sum_{k=1}^{\infty} e^{-k^2\pi^2 t} \int_{-1}^1 \tilde{a}_0(y) \cos(k\pi(x-y))\, dy \quad (76)$$
$$+2\int_0^t \int_0^1 g(s,y)\, dyds + \sum_{k=1}^{\infty} \int_0^t e^{-k^2\pi^2(t-s)} \int_{-1}^1 \tilde{g}(s,y) \cos(k\pi(x-y))\, dyds\,.$$

With the Fourier transform normalised as $\hat{f}(k) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(y) e^{iky}\, dy$, we invert $\hat{f}(k) = e^{-k^2\pi^2 t}$ to $f(x) = (2t)^{-\frac{1}{2}} \pi^{-1} e^{-x^2/(4\pi^2 t)}$. Then, using the Poisson summation formula $\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikx} = \sqrt{2\pi} \sum_{k=-\infty}^{\infty} f(x + 2k\pi)$ and since $\hat{f}(k)$ is even, we obtain that

$$1 + 2\sum_{k=1}^{\infty} e^{-k^2\pi^2 t} \cos(k\pi(x-y)) = \frac{1}{\sqrt{\pi t}} \sum_{k=-\infty}^{\infty} e^{-\frac{(2k+x-y)^2}{4t}}\,. \quad (77)$$

Thus, by (76) and (77),

$$\tilde{a}(t,x) = \int_0^1 a_0(y)\,dy + \tfrac{1}{2\sqrt{\pi}}\int_{-1}^1 \tilde{a_0}(y)\sum_{k=-\infty}^{\infty}\tfrac{1}{\sqrt{t}}e^{-\frac{(2k+x-y)^2}{4t}}dy \tag{78}$$
$$+\int_0^t\int_0^1 g(s,y)\,dyds + \tfrac{1}{2\sqrt{\pi}}\int_0^t\int_{-1}^1 \tilde{g}(s,y)\sum_{k=-\infty}^{\infty}\tfrac{1}{\sqrt{t-s}}e^{-\frac{(2k+x-y)^2}{4(t-s)}}dyds\,.$$

We define then $S(t,x) := \sum_{k=-\infty}^{\infty}\frac{1}{\sqrt{t}}e^{-\frac{(2k+x)^2}{4t}}$ and rewrite and estimate

$$\|\tilde{a}(t,x)\|_{L^r([0,T]\times[-1,1])} \le (2T)^{\frac{1}{r}}\|a_0\|_{L^1([0,1])} + \tfrac{1}{2\sqrt{\pi}}\|\tilde{a_0} *_x S\|_{L^r([0,T]\times[-1,1])}$$
$$+(2T)^{\frac{1}{r}}\|g\|_{L^1([0,T]\times[0,1])} + \tfrac{1}{2\sqrt{\pi}}\|\tilde{g} *_{t,x} S\|_{L^r([0,T]\times[-1,1])}\,, \tag{79}$$

where $*_x$ and $*_{t,x}$ denote the convolution in x and t,x respectively. For the function $S := S(t,x)$, we prove the lemma:

Lemma 3.4 *Let $S(t,x) = \sum_{k=-\infty}^{\infty}\frac{1}{\sqrt{t}}e^{-\frac{(2k+x)^2}{4t}}$. Then,*

$$\begin{aligned}\|S(t,\cdot)\|_{L^1([-1,1])} &= 2\sqrt{\pi}\,,\\ \|S\|_{L^q([0,T]\times[-1,1])} &< K_{18}(q)\left(1+T^{\frac{1}{q}+\frac{1}{2}}\right), \qquad q\in[1,3)\,.\end{aligned}$$

Proof: The first statement is easily obtained by

$$\frac{1}{\sqrt{t}}\sum_{k=-\infty}^{\infty}\int_{-1}^{1}e^{-\frac{(2k+x)^2}{4t}}\,dx = \frac{1}{\sqrt{t}}\int_{-\infty}^{\infty}e^{-\frac{x^2}{4t}}\,dx = 2\sqrt{\pi}\,.$$

For the second, we consider $q\in(1,3)$ and use (when $n\neq 0$) $(2n+x)^2 \ge |2n+x| \ge 2n-1$ to estimate

$$\|S\|_{L^q([0,T]\times[-1,1])} = \left\|t^{-\frac{1}{2}}\left(e^{-\frac{x^2}{4t}} + 2\sum_{n=1}^{\infty}e^{-\frac{(2n+x)^2}{4t}}\right)\right\|_{L^q([0,T]\times[-1,1])}$$
$$\le \left\|t^{-\frac{1}{2}}\left(e^{-\frac{x^2}{4t}} + 2\sum_{n=1}^{\infty}e^{-\frac{2n-1}{4t}}\right)\right\|_{L^q([0,T]\times[-1,1])}\,,$$

and further, using the geometric sum $\sum_{n=1}^{\infty}e^{-(2n-1)/4t} = (e^{1/4t}-e^{-1/4t})^{-1}$,

$$\le \left\|t^{-\frac{1}{2}}e^{-\frac{x^2}{4t}}\right\|_{L^q([0,T]\times[-1,1])} + 2\left\|t^{-\frac{1}{2}}(e^{1/4t}-e^{-1/4t})^{-1}\right\|_{L^q([0,T]\times[-1,1])}\,.$$

Then, thanks to a Taylor expansion, the second term writes (for some $\xi\in[0,\frac{1}{4t}]$)

$$(e^{1/4t}-e^{-1/4t})^{-1} = 2t\left(1+\frac{e^{\xi}+e^{-\xi}}{192t^2}\right)^{-1} \le 2t\,.$$

Therefore, noting that $\int_{-1}^{1} e^{-\frac{x^2 q}{4t}}\,dx = 2\sqrt{\pi}\sqrt{\frac{t}{q}}\,\mathrm{erf}\left(\frac{\sqrt{q}}{2\sqrt{t}}\right)$, we obtain for $q \in (1,3)$

$$\begin{aligned}
&\leq 2^{\frac{1}{q}} \left(\tfrac{\pi}{q}\right)^{\frac{1}{2q}} \left(\int_0^T t^{\frac{1-q}{2}}\,dt\right)^{\frac{1}{q}} + 2^{2+\frac{1}{q}} \left(\int_0^T t^{\frac{q}{2}}\,dt\right)^{\frac{1}{q}} \\
&= 2^{\frac{1}{q}} \left(\tfrac{\pi}{q}\right)^{\frac{1}{2q}} \left(\tfrac{2}{3-q}\right)^{\frac{1}{q}} T^{\frac{3-q}{2q}} + 2^{2+\frac{1}{q}} \left(\tfrac{2}{2+q}\right)^{\frac{1}{q}} T^{\frac{1}{q}+\frac{1}{2}}. \qquad (80)
\end{aligned}$$

Moreover, since $\frac{3-q}{2q} < \frac{1}{q} + \frac{1}{2}$ for $q \in (1,3)$, we denote by T_0 the time which equals the two terms of (80) and estimate

$$\begin{aligned}
(80) &= 2^{\frac{1}{q}} \left(\frac{\pi}{q}\right)^{\frac{1}{2q}} \left(\frac{2}{3-q}\right)^{\frac{1}{q}} T_0^{\frac{3-q}{2q}} \left(\left(\frac{T}{T_0}\right)^{\frac{3-q}{2q}} + \left(\frac{T}{T_0}\right)^{\frac{1}{q}+\frac{1}{2}}\right) \\
&< 2^{\frac{1}{q}} \left(\frac{\pi}{q}\right)^{\frac{1}{2q}} \left(\frac{2}{3-q}\right)^{\frac{1}{q}} T_0^{\frac{3-q}{2q}} 2 \left(1 + T_0^{-\frac{1}{q}-\frac{1}{2}}\right) \left(1 + T^{\frac{1}{q}+\frac{1}{2}}\right) \\
&= K_{18}(q)\,(1 + T^{\frac{1}{q}+\frac{1}{2}}). \qquad (81)
\end{aligned}$$

For $q = 1$, we have explicitly $\|S\|_{L^1([0,T]\times[-1,1])} = 2\sqrt{\pi}\,T < K_{18}(1 + T^{\frac{1}{q}+\frac{1}{2}})$. □

Continuation of the proof of lemma 3.3:
Returning to (79), we estimate the second term on the right hand side by

$$\begin{aligned}
\frac{1}{2\sqrt{\pi}} \|\tilde{a}_0 *_x S(t,\cdot)\|_{L^r([0,T]\times[-1,1])} &\leq \frac{2^{\frac{1}{r}}}{2\sqrt{\pi}} \left(\int_0^T \|\tilde{a}_0 *_x S(t,\cdot)\|_{L^\infty([-1,1])}\,dt\right)^{\frac{1}{r}} \\
&\leq (2T)^{\frac{1}{r}} \|a_0\|_{L^\infty([0,1])}, \qquad (82)
\end{aligned}$$

since $\|\tilde{a}_0 *_x S(t,\cdot)\|_{L^\infty([-1,1])} \leq \|\tilde{a}_0\|_{L^\infty([-1,1])} \|S(t,\cdot)\|_{L^1([-1,1])}$ and thanks to lemma 3.4.

Finally, for the fourth term on the right hand side of (79), we apply Young's inequality $\|\tilde{g} * S\|_{L^r} \leq \|\tilde{g}\|_{L^p} \|S\|_{L^q}$ for $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$ and lemma 3.4. Collecting (79) with (82) and (81), we obtain (72) since $\|\tilde{a}\|_{L^r([0,T]\times[-1,1])} = 2^{\frac{1}{r}} \|a\|_{L^r([0,T]\times[0,1])}$. □

Combining lemma 3.2 with lemma 3.3 shows in three iterations that the L^∞-norm of 1D solutions of (19)-(25) increases at most polynomially in time: this is detailed in the following

Lemma 3.5 *Let a_i, $i = 1,2,3,4$ be solutions of the system (19)-(25) with bounded initial data $a_{i,0} \in L^\infty(\Omega)$. Then, for $T > 0$*

$$\|a_i\|_{L^\infty([0,T]\times[0,1])} \leq K_{19,i}\left(1 + T^{\frac{21}{2}}\right),$$

with the constant

$$K_{19,i} = \|a_{i,0}\|_{L^\infty[0,1]} + \|a_{i,0}\|_{L^1[0,1]} + \sum_{i=1}^{4} K_{17,i} + 3\, K_{18} \sum_{i=1}^{4} K_{21,i}^2\,, \qquad (83)$$

depending on the constants $K_{17,i}$ (66), K_{18} (73), $K_{21,i}$ (88), and K_{20} (86).

Proof: We apply the following bootstrap. By lemma 3.2, we have

$$\|ab - cd\|_{L^1([0,T]\times[0,1])} \le \frac{1}{2} \sum_{i=1}^{4} K_{17,i}\,(1+T)\,,$$

and moreover by lemma 3.3 with $p = 1$ and $r = q \in [1,3)$ and $i = 1,2,3,4$

$$\begin{aligned} \|a_i\|_{L^r([0,T]\times[0,1])} &\le T^{\frac{1}{r}} \left(\|a_{i,0}\|_{L^\infty[0,1]} + \|a_{i,0}\|_{L^1[0,1]} + \frac{1}{2}\sum_{i=1}^{4} K_{17,i}\,(1+T) \right) \\ &\quad + K_{18}\, \frac{1}{2}\sum_{i=1}^{4} K_{17,i}\,(1 + T^{\frac{1}{r}+\frac{1}{2}})(1+T)\,, \qquad (84) \\ &< (\|a_{i,0}\|_{L^\infty[0,1]} + \|a_{i,0}\|_{L^1[0,1]} + \frac{3}{2}K_{18}\sum_{i=1}^{4} K_{17,i})(1 + T^{\frac{1}{r}+\frac{3}{2}})\,, \end{aligned}$$

since $K_{18} > 1$. In a second step, we apply again lemma 3.2 for $s \in [2,3)$ and obtain by (84) that

$$\begin{aligned} \|ab - cd\|_{L^{\frac{s}{2}}([0,T]\times[0,1])} &\le \frac{1}{2} \left\| \sum_{i=1}^{4} a_i^2 \right\|_{L^{\frac{s}{2}}([0,T]\times[0,1])} \le \frac{1}{2}\sum_{i=1}^{4} \|a_i^2\|_{L^{\frac{s}{2}}([0,T]\times[0,1])} \\ &= \frac{1}{2}\sum_{i=1}^{4} \|a_i\|^2_{L^s([0,T]\times[0,1])} < K_{20}\,(1 + T^{\frac{2}{s}+3})\,, \qquad (85) \end{aligned}$$

with

$$K_{20} = \sum_{i=1}^{4} \left(\|a_{i,0}\|_{L^\infty[0,1]} + \|a_{i,0}\|_{L^1[0,1]} + \frac{3}{2}K_{18} \sum_{i=1}^{4} K_{17,i} \right)^2 . \qquad (86)$$

Then, by lemma 3.3 with $p = \frac{s}{2}$, $q \in [1,3)$ and $r \in [1,\infty)$, it follows that

$$\begin{aligned} \|a_i\|_{L^r([0,T]\times[0,1])} &< T^{\frac{1}{r}}(\|a_{i,0}\|_{L^\infty[0,1]} + \|a_{i,0}\|_{L^1[0,1]} + \frac{1}{2}\sum_{i=1}^{4} K_{17,i}(1+T)) \\ &\quad + K_{18}\, K_{20}\,(1 + T^{\frac{1}{q}+\frac{1}{2}})(1 + T^{\frac{2}{s}+3}) \\ &< K_{21,i}(1 + T^{\frac{1}{r}+\frac{9}{2}})\,, \qquad (87) \end{aligned}$$

since $T^{\frac{1}{q}+\frac{1}{2}}T^{\frac{2}{s}+3} = T^{\frac{1}{r}+\frac{9}{2}}$ and with the constants

$$K_{21,i} = \|a_{i,0}\|_{L^\infty[0,1]} + \|a_{i,0}\|_{L^1[0,1]} + \sum_{i=1}^{4} K_{17,i} + 3\,K_{18}\,K_{20}\,. \tag{88}$$

In a final step, we observe firstly (similar to (85) for $s \in [2,\infty)$) that

$$\|ab-cd\|_{L^{\frac{s}{2}}([0,T]\times[0,1])} \le \frac{1}{2}\sum_{i=1}^{4}\|a_i\|^2_{L^s([0,T]\times[0,1])} < \sum_{i=1}^{4} K^2_{21,i}\,(1+T^{\frac{2}{s}+9})\,, \tag{89}$$

and, secondly, by lemma 3.3 with $p=\frac{s}{2}$, $r=\infty$, and $1=\frac{1}{p}+\frac{1}{q}$, that

$$\begin{aligned}\|a_i\|_{L^\infty([0,T]\times[0,1])} &< \|a_{i,0}\|_{L^\infty[0,1]} + \|a_{i,0}\|_{L^1[0,1]} + \frac{1}{2}\sum_{i=1}^{4} K_{17,i}(1+T)\\ &\quad + K_{18}\sum_{i=1}^{4} K^2_{21,i}\,(1+T^{\frac{1}{q}+\frac{1}{2}})(1+T^{\frac{2}{s}+9})\\ &< K_{19,i}\,(1+T^{\frac{21}{2}})\,,\end{aligned}$$

since $T^{\frac{1}{q}+\frac{1}{2}}T^{\frac{2}{s}+9} = T^{\frac{21}{2}}$ and with the constant $K_{19,i}$ define in (83). □

3.2 Convergence

The proof of theorem 1.2 is divided into six lemmas. Lemma 3.6 shows, for the special situation where A, B, C, and D are constants subject to the conservation laws (26)– (28), how L^2-distances with respect to the steady state are controlled by a L^2-distance coming from the reaction:

Lemma 3.6 *Let A_∞, B_∞, C_∞, and D_∞ denote the positive square roots of (33). Let $\overline{A}$, $\overline{B}$, $\overline{C}$, and $\overline{D}$ be nonnegative constants satisfying the conservation laws (26) – (28), i.e. $\overline{A}^2+\overline{C}^2 = A^2_\infty + C^2_\infty$, $\overline{A}^2+\overline{D}^2 = A^2_\infty + D^2_\infty$, $\overline{B}^2+\overline{C}^2 = B^2_\infty + C^2_\infty$. Then,*

$$\|\overline{A}-A_\infty\|_2^2+\|\overline{B}-B_\infty\|_2^2+\|\overline{C}-C_\infty\|_2^2+\|\overline{D}-D_\infty\|_2^2 \le K_9\|\overline{A}\,\overline{B}-\overline{C}\,\overline{D}\|_2^2\,, \tag{90}$$

where

$$K_9 = \frac{1+\max\{4\frac{A^2_\infty}{B^2_\infty},1\}}{B^2_\infty} + \left(\frac{1}{C^2_\infty}+\frac{1}{D^2_\infty}\right)\frac{\left(A_\infty+\sqrt{A_\infty+\min\{C^2_\infty,D^2_\infty\}}\right)^2}{B^2_\infty}\,. \tag{91}$$

Proof of lemma 3.6: The proof is based on the ansatz

$$\overline{A_i}^2 = A_{i,\infty}^2 (1+\mu_i)^2 \,, \qquad -1 \le \mu_i \,, \quad \text{for } i = 1,2,3,4 \,. \tag{92}$$

The conservation laws (26)–(28), and more precisely the relations

$$A_\infty^2(2\mu_1 + \mu_1^2) - B_\infty^2(2\mu_2 + \mu_2^2) = 0 \,, \tag{93}$$
$$A_\infty^2(2\mu_1 + \mu_1^2) + C_\infty^2(2\mu_3 + \mu_3^2) = 0 \,, \tag{94}$$
$$A_\infty^2(2\mu_1 + \mu_1^2) + D_\infty^2(2\mu_4 + \mu_4^2) = 0 \,, \tag{95}$$

allow to express μ_2, μ_3, and μ_4 as function of μ_1 as stated in the following lemma:

Lemma 3.7 ***(Ranges of*** μ_1***,*** μ_2***,*** μ_3***, and*** μ_4***)***
Let $\mu_2(\mu_1), \mu_3(\mu_1), \mu_4(\mu_1)$ *be defined via (93)–(95), i.e.*

$$\mu_2(\mu_1) = -1 + \sqrt{1 + \tfrac{A_\infty^2}{B_\infty^2}(2\mu_1 + \mu_1^2)} \,, \tag{96}$$

$$\mu_3(\mu_1) = -1 + \sqrt{1 - \tfrac{A_\infty^2}{C_\infty^2}(2\mu_1 + \mu_1^2)} \,, \tag{97}$$

$$\mu_4(\mu_1) = -1 + \sqrt{1 - \tfrac{A_\infty^2}{D_\infty^2}(2\mu_1 + \mu_1^2)} \,. \tag{98}$$

Then, $\mu_2(\mu_1)$ *is monotone increasing in* μ_1*, while* $\mu_3(\mu_1)$ *and* $\mu_4(\mu_1)$ *are monotone decreasing in* μ_1*. Finally,* $\mu_i(\mu_1) = 0$ *if and only if* $\mu_1 = 0$ *for* $i = 2,3,4$*.*

Moreover, since we know $\mu_2(\mu_1)$*,* $\mu_3(\mu_1)$*,* $\mu_4(\mu_1)$ *to be real,* μ_1 *restricts to*

$$\mu_{1,min} \le \mu_1 \le \mu_{1,max} \,, \tag{99}$$

with

$$\mu_{1,min} = -1 + \sqrt{1 - \tfrac{\min\{A_\infty^2, B_\infty^2\}}{A_\infty^2}} \,, \ \mu_{1,max} = -1 + \sqrt{1 + \tfrac{\min\{C_\infty^2, D_\infty^2\}}{A_\infty^2}} \,, \tag{100}$$

implying by monotonicity of $\mu_2(\mu_1)$*,* $\mu_3(\mu_1)$*, and* $\mu_4(\mu_1)$ *the admissible ranges*

$$-1 \le \mu_2(\mu_{1,min}) \le \mu_2(0) = 0 \le \mu_2(\mu_{1,max}) \,, \tag{101}$$
$$-1 \le \mu_3(\mu_{1,max}) \le \mu_3(0) = 0 \le \mu_3(\mu_{1,min}) \,, \tag{102}$$
$$-1 \le \mu_4(\mu_{1,max}) \le \mu_4(0) = 0 \le \mu_4(\mu_{1,min}) \,. \tag{103}$$

The next lemma quantifies how $\mu_i(\mu_1)$ for $i = 2,3,4$ is proportional to μ_1:

Lemma 3.8 ***(Taylor expansion of*** μ_2***,*** μ_3***, and*** μ_4***)***
A Taylor-expansion of (96), (97), and (98) yields

$$\mu_2 = \tfrac{A_\infty}{B_\infty^2} R_2(\mu_1)\mu_1 \,, \quad \mu_3 = -\tfrac{A_\infty}{C_\infty^2} R_3(\mu_1)\mu_1 \,, \quad \mu_4 = -\tfrac{A_\infty}{D_\infty^2} R_4(\mu_1)\mu_1 \,, \tag{104}$$

where $R_2(\mu_1)$*,* $R_3(\mu_1)$*, and* $R_4(\mu_1)$ *are strictly positive for* $\mu_1 \in [\mu_{1,min}, \mu_{1,max}]$ *and bounded by*

$$0 < R_2 \;\leq\; \max\{2A_\infty, B_\infty\} \,, \tag{105}$$

$$0 < R_3, R_4 \;\leq\; A_\infty + \sqrt{A_\infty^2 + \min\{C_\infty^2, D_\infty^2\}} \,. \tag{106}$$

Proof: We Taylor-expand

$$\sqrt{1 + \tfrac{A_\infty^2}{B_\infty^2}(2\mu_1 + \mu_1^2)} = 1 + \frac{1+\zeta}{\sqrt{1+\frac{A_\infty^2}{B_\infty^2}(2\zeta+\zeta^2)}} \frac{A_\infty^2}{B_\infty^2}\mu_1 \,,$$

for some $\zeta \in (0, \mu_1)$ and consider the remainder

$$R_2(\mu_1) = \frac{A_\infty(1+\zeta)}{\sqrt{1+\frac{A_\infty^2}{B_\infty^2}(2\zeta+\zeta^2)}} = \frac{B_\infty^2}{A_\infty}\frac{1}{\mu_1}\left(-1 + \sqrt{1 + \tfrac{A_\infty^2}{B_\infty^2}(2\mu_1 + \mu_1^2)}\right) \,,$$

for $\mu_1 \in [\mu_{1,min}, \mu_{1,max}]$. It is straightforward to show that $R_2(\mu_1)$ is monotone increasing or decreasing in $\mu_1 \in [\mu_{1,min}, \mu_{1,max}]$ if and only if $A_\infty < B_\infty$ or $A_\infty > B_\infty$, respectively, and continuous at $\mu_1 = 0$ with $R_2(0) = A_\infty$. Therefore,

$$0 < R_2(\mu_{1,min}) < R_2(\mu_{1,max}) = \frac{A_\infty + \sqrt{A_\infty + \min\{C_\infty^2, D_\infty^2\}}}{1+\sqrt{1+\frac{\min\{C_\infty^2, D_\infty^2\}}{B_\infty^2}}} \leq B_\infty \,, \quad A_\infty \leq B_\infty \,,$$

$$0 < R_2(\mu_{1,max}) < R_2(\mu_{1,min}) = A_\infty + \sqrt{A_\infty^2 - B_\infty^2} < 2A_\infty \,, \quad A_\infty \geq B_\infty \,,$$

which yields (105).

For μ_3 (and analogously for μ_4), we Taylor-expand

$$\sqrt{1 - \tfrac{A_\infty^2}{C_\infty^2}(2\mu_1 + \mu_1^2)} = 1 - \frac{1+\zeta}{\sqrt{1-\frac{A_\infty^2}{C_\infty^2}(2\zeta+\zeta^2)}} \frac{A_\infty^2}{C_\infty^2}\mu_1 \,,$$

for some $\zeta \in (0, \mu_1)$ and consider the remainder

$$R_3(\mu_1) = \frac{A_\infty(1+\zeta)}{\sqrt{1-\frac{A_\infty^2}{C_\infty^2}(2\zeta+\zeta^2)}} = -\frac{C_\infty^2}{A_\infty}\frac{1}{\mu_1}\left(1 - \sqrt{1 - \tfrac{A_\infty^2}{C_\infty^2}(2\mu_1 + \mu_1^2)}\right) \,,$$

which increases with respect to μ_1, and $R_3(0) = A_\infty$ and

$$0 < R_3(\mu_{1,min}) < R_3(\mu_{1,max}) \leq A_\infty + \sqrt{A_\infty^2 + \min\{C_\infty^2, D_\infty^2\}} \,.$$

□

Continuation of the proof of lemma 3.6
Using the ansatz (92) to expand (90) (and with $A_\infty B_\infty = C_\infty D_\infty$), we have to prove that

$$\frac{A_\infty^2\mu_1^2 + B_\infty^2\mu_2^2 + C_\infty^2\mu_3^2 + D_\infty^2\mu_4^2}{A_\infty^2 B_\infty^2\left(\mu_1+\mu_2+\mu_1\mu_2-\mu_3-\mu_4-\mu_3\mu_4\right)^2} \leq K_9\,, \tag{107}$$

for $\mu_1 \in [\mu_{1,min}, \mu_{1,max}]$ and with $\mu_2(\mu_1), \mu_3(\mu_1), \mu_4(\mu_1)$ as in lemma 3.7.
Considering the numerator of (107), we estimate with lemma 3.8 that

$$\begin{aligned} A_\infty^2\mu_1^2 + B_\infty^2\mu_2^2 + C_\infty^2\mu_3^2 + D_\infty^2\mu_4^2 &\leq \mu_1^2 A_\infty^2\left(1+\frac{R_2^2}{B_\infty^2}+\frac{R_3^2}{C_\infty^2}+\frac{R_4^2}{D_\infty^2}\right) \\ &\leq \mu_1^2 A_\infty^2 B_\infty^2 K_9\,. \end{aligned} \tag{108}$$

Concerning the denominator of (107), we observe first by lemma 3.7 that assuming $\mu_1 < 0$ in the sum $\mu_1+\mu_2+\mu_1\mu_2+(-\mu_3)+(-\mu_4)+(-\mu_3\mu_4)$, only the term $\mu_1\mu_2$ is non-negative and all the other terms are non-positive. Moreover, we know in this case that $-1 \leq \mu_1$ and $-1 \leq \mu_2$ and therefore $\mu_2 \leq -\mu_1\mu_2$ implying

$$\mu_1+\mu_2+\mu_1\mu_2-\mu_3-\mu_4-\mu_3\mu_4 \leq \mu_1-\mu_3-\mu_4-\mu_3\mu_4 \leq -|\mu_1|\,. \tag{109}$$

If we now assume that $\mu_1 > 0$, only the term $-\mu_3\mu_4$ is non-positive and $-1 \leq \mu_3$ as well as $-1 \leq \mu_4$, therefore $\mu_3 \leq -\mu_3\mu_4$ and

$$\mu_1+\mu_2+\mu_1\mu_2-\mu_3-\mu_4-\mu_3\mu_4 \geq \mu_1+\mu_2+\mu_1\mu_2-\mu_4 \geq |\mu_1|\,. \tag{110}$$

Altogether, by (109) and (110), we estimate the denominator of (107) by

$$A_\infty^2 B_\infty^2\left(\mu_1+\mu_2+\mu_1\mu_2-\mu_3-\mu_4-\mu_3\mu_4\right)^2 \geq A_\infty^2 B_\infty^2 \mu_1^2\,,$$

which proves (with (108)) that we can take the constant (91), and lemma 3.6 is obtained. □

The following lemma extends lemma 3.6 to nonnegative functions A, B, C, and D, which satisfy the conservation laws (26) – (28).

Lemma 3.9 *Let A_∞, B_∞, C_∞, and D_∞ denote the positive square roots of (33). Let A, B, C, and D be measurable, nonnegative functions satisfying the conservation laws (26) – (28), i.e. $\overline{A^2}+\overline{C^2} = M_{13} = A_\infty^2 + C_\infty^2$, $\overline{A^2}+\overline{D^2} = M_{14} = A_\infty^2 + D_\infty^2$, $\overline{B^2}+\overline{C^2} = M_{23} = B_\infty^2 + C_\infty^2$. Then,*

$$\begin{aligned} \|A-A_\infty\|_2^2 + \|B-B_\infty\|_2^2 + \|C-C_\infty\|_2^2 + \|D-D_\infty\|_2^2 &\leq K_{10}\|AB-CD\|_2^2 \\ +K_{11}(\|A-\overline{A}\|_2^2 + \|B-\overline{B}\|_2^2 + \|C-\overline{C}\|_2^2 &+ \|D-\overline{D}\|_2^2)\,, \end{aligned} \tag{111}$$

where, recalling (31), the total mass is $M_{1234} = A_\infty^2 + B_\infty^2 + C_\infty^2 + D_\infty^2$*, and*

$$K_{10} = \max\{K_9, \max_{i=1,2}\left\{\frac{4M_{1234}}{M_{i3}M_{i4}}\right\}, \max_{i=3,4}\left\{\frac{4M_{1234}}{M_{1i}M_{2i}}\right\}\}, \tag{112}$$

with K_9 *defined in (91) and*

$$K_{11} = \begin{cases} K_{10}\left(\sqrt{M_{14}M_{23}} + M_{1234}\right), & \text{if } \sqrt{\overline{A_i^2}} \le \varepsilon_i \text{ for some } i = 1,2,3,4\,, \\ K_9\sqrt{M_{14}M_{23}}\left(1 + \max\limits_{i=1,2,3,4}\left\{\frac{2\sqrt{M_{1234}}}{\varepsilon_i}\right\}\right) + \max\limits_{i=1,2,3,4}\left\{\frac{2A_{i,\infty}}{\varepsilon_i}\right\}, & \text{else} \end{cases} \tag{113}$$

where

$$\varepsilon_i = \frac{\sqrt{M_{1234}\,M_{i3}\,M_{i4}}}{M_{i3}+M_{i4}}\left(\sqrt{1+\frac{M_{i3}+M_{i4}}{2M_{1234}}} - 1\right), \quad i = 1,2\,, \tag{114}$$

$$\varepsilon_i = \frac{\sqrt{M_{1234}\,M_{1i}\,M_{2i}}}{M_{1i}+M_{2i}}\left(\sqrt{1+\frac{M_{1i}+M_{2i}}{2M_{1234}}} - 1\right), \quad i = 3,4\,. \tag{115}$$

Proof: Since we want to apply lemma 3.6, we expand around the mean values

$$A_i = \overline{A_i} + \delta_i(x)\,, \qquad \overline{\delta_i} = 0\,, \quad \text{for } i = 1,2,3,4\,, \tag{116}$$

and modify then the ansatz of lemma 3.6 into

$$\overline{A_i^2} = A_{i,\infty}^2(1+\mu_i)^2\,, \qquad -1 \le \mu_i \quad \text{for } i = 1,2,3,4\,, \tag{117}$$

in order to preserve the relations (93)-(95) and therefore the ranges of μ_1 and $\mu_i(\mu_1)$, $i = 2,3,4$ as stated in lemma 3.7 together with the Taylor expansions in lemma 3.8.

The ansatz (116), (117) implies readily for the right hand side of (111) that

$$\|A_i - \overline{A_i}\|_2^2 = \overline{A_i^2} - \overline{A_i}^2 = \overline{\delta_i^2}\,, \qquad \text{for } i = 1,2,3,4\,. \tag{118}$$

Since

$$\frac{\overline{\delta_i^2}}{\sqrt{\overline{A_i^2}} + \overline{A_i}} = \sqrt{\overline{A_i^2}} - \overline{A_i}\,, \qquad \text{for } i = 1,2,3,4\,, \tag{119}$$

it follows that

$$\overline{A_i} = A_{i,\infty}(1+\mu_i) - \frac{1}{\sqrt{\overline{A_i^2}} + \overline{A_i}}\,\overline{\delta_i^2}\,, \qquad \text{for } i = 1,2,3,4\,. \tag{120}$$

Thus the expansions in terms of $\overline{\delta_i^2}$ is unbounded for vanishing $\overline{A_i^2}$ and the same for the left hand side of (111), where we use (120) to expand

$$\|A_i - A_{i,\infty}\|_2^2 = A_{i,\infty}^2 \mu_i^2 + \frac{2A_{i,\infty}}{\sqrt{\overline{A_i^2}} + \overline{A_i}} \overline{\delta_i^2}, \qquad \text{for } i = 1,2,3,4. \qquad (121)$$

Leaving the singularity for small $\overline{A_i^2}$ for later, we factorise in a first step

$$\begin{aligned} \|AB - CD\|_2^2 &= \|\overline{A}\,\overline{B} - \overline{C}\,\overline{D}\|_2^2 + 2(\overline{A}\,\overline{B} - \overline{C}\,\overline{D})(\overline{\delta_1\delta_2} - \overline{\delta_3\delta_4}) \\ &\quad + \|\overline{A}\delta_2 + \overline{B}\delta_1 + \delta_1\delta_2 - \overline{C}\delta_4 - \overline{D}\delta_3 - \delta_3\delta_4\|_2^2. \quad (122) \end{aligned}$$

Since $\overline{A_i} \leq \sqrt{\overline{A_i^2}}$ by Jensen's inequality and $\overline{A^2}\,\overline{B^2} \leq M_{14}M_{23}$, $\overline{C^2}\,\overline{D^2} \leq M_{14}M_{23}$ by the conservation laws (26)–(29), we estimate the second term on the right hand side of (122) using Young's inequality:

$$\begin{aligned} 2(\overline{A}\,\overline{B} - \overline{C}\,\overline{D})(\overline{\delta_1\delta_2} - \overline{\delta_3\delta_4}) &\geq -|\overline{A}\,\overline{B} - \overline{C}\,\overline{D}|\,(\overline{\delta_1^2} + \overline{\delta_2^2} + \overline{\delta_3^2} + \overline{\delta_4^2}) \\ &\geq -\sqrt{M_{14}M_{23}}\,(\overline{\delta_1^2} + \overline{\delta_2^2} + \overline{\delta_3^2} + \overline{\delta_4^2}). \quad (123) \end{aligned}$$

Then, we insert (120) (recalling $A_\infty B_\infty = C_\infty D_\infty$) into

$$\begin{aligned} \|\overline{A}\,\overline{B} - \overline{C}\,\overline{D}\|_2^2 &= A_\infty^2 B_\infty^2 (\mu_1 + \mu_2 + \mu_1\mu_2 - \mu_3 - \mu_4 - \mu_3\mu_4) \\ &\quad -2\left(\sqrt{\overline{A^2}\,\overline{B^2}} - \sqrt{\overline{C^2}\,\overline{D^2}}\right)\left(\frac{\sqrt{\overline{B^2}}\,\overline{\delta_1^2}}{\sqrt{\overline{A^2}} + \overline{A}} + \frac{\sqrt{\overline{A^2}}\,\overline{\delta_2^2}}{\sqrt{\overline{B^2}} + \overline{B}}\right. \\ &\quad -\frac{\overline{\delta_1^2}\,\overline{\delta_2^2}}{(\sqrt{\overline{A^2}} + \overline{A})(\sqrt{\overline{B^2}} + \overline{B})} - \frac{\sqrt{\overline{D^2}}\,\overline{\delta_3^2}}{\sqrt{\overline{C^2}} + \overline{C}} \\ &\quad \left. -\frac{\sqrt{\overline{C^2}}\,\overline{\delta_4^2}}{\sqrt{\overline{D^2}} + \overline{D}} + \frac{\overline{\delta_3^2}\,\overline{\delta_4^2}}{(\sqrt{\overline{C^2}} + \overline{C})(\sqrt{\overline{D^2}} + \overline{D})}\right) \\ &\quad + \left\| \frac{\sqrt{\overline{B^2}}\,\overline{\delta_1^2}}{\sqrt{\overline{A^2}} + \overline{A}} + \ldots + \frac{\overline{\delta_3^2}\,\overline{\delta_4^2}}{(\sqrt{\overline{C^2}} + \overline{C})(\sqrt{\overline{D^2}} + \overline{D})} \right\|_2^2. \quad (124) \end{aligned}$$

For the second factor on the right hand side of (124), we estimate similar to above $\left|\sqrt{\overline{A^2}\,\overline{B^2}} - \sqrt{\overline{C^2}\,\overline{D^2}}\right| \leq \sqrt{M_{14}M_{23}}$ and use (119) to calculate

$$\frac{\sqrt{\overline{B^2}}\,\overline{\delta_1^2}}{\sqrt{\overline{A^2}}+\overline{A}} + \frac{\sqrt{\overline{A^2}}\,\overline{\delta_2^2}}{\sqrt{\overline{B^2}}+\overline{B}} - \frac{\overline{\delta_1^2}\,\overline{\delta_2^2}}{(\sqrt{\overline{A^2}}+\overline{A})(\sqrt{\overline{B^2}}+\overline{B})} = \frac{1}{2}\frac{\sqrt{\overline{B^2}}+\overline{B}}{\sqrt{\overline{A^2}}+\overline{A}}\overline{\delta_1^2} + \frac{1}{2}\frac{\sqrt{\overline{A^2}}+\overline{A}}{\sqrt{\overline{B^2}}+\overline{B}}\overline{\delta_2^2}$$

and we compute in the same way the product proportional to $\overline{\delta_3^2\delta_4^2}$. Thus, by $\overline{A_i} \leq \sqrt{\overline{A_i^2}} < \sqrt{M_{1234}}$ for all $i = 1,2,3,4$, we obtain

$$\begin{aligned}
&-\sqrt{M_{14}M_{23}}\left|\frac{\sqrt{\overline{B^2}}+\overline{B}}{\sqrt{\overline{A^2}}+\overline{A}}\overline{\delta_1^2} + \frac{\sqrt{\overline{A^2}}+\overline{A}}{\sqrt{\overline{B^2}}+\overline{B}}\overline{\delta_2^2} - \frac{\sqrt{\overline{D^2}}+\overline{D}}{\sqrt{\overline{C^2}}+\overline{C}}\overline{\delta_3^2} - \frac{\sqrt{\overline{C^2}}+\overline{C}}{\sqrt{\overline{D^2}}+\overline{D}}\overline{\delta_4^2}\right| \\
&\quad \geq -\sqrt{M_{14}M_{23}} \max_{i=1,2,3,4}\left\{\frac{2\sqrt{M_{1234}}}{\sqrt{\overline{A_i^2}}}\right\}\left(\overline{\delta_1^2}+\overline{\delta_2^2}+\overline{\delta_3^2}+\overline{\delta_4^2}\right). \qquad (125)
\end{aligned}$$

Thus, inserting (121) into the left hand side of (111) and combining (118) and (122)–(125) for the right hand side of (111) we have to prove that

$$\begin{aligned}
\sum_{i=1}^{4} A_{i,\infty}^2\mu_i^2 \;\leq\; & K_{10}A_\infty^2 B_\infty^2\left(\mu_1+\mu_2+\mu_1\mu_2-\mu_3-\mu_4-\mu_3\mu_4\right)^2 \\
& + \left(K_{11} - K_{10}\sqrt{M_{14}M_{23}}\left(1+\max_{i=1,2,3,4}\left\{\frac{2\sqrt{M_{1234}}}{\sqrt{\overline{A_i^2}}}\right\}\right)\right. \\
& \left. - \max_{i=1,2,3,4}\left\{\frac{2A_{i,\infty}}{\sqrt{\overline{A_i^2}}+\overline{A_i}}\right\}\right)\left(\overline{\delta_1^2}+\overline{\delta_2^2}+\overline{\delta_3^2}+\overline{\delta_4^2}\right).
\end{aligned}$$

When $K_{10} \geq K_9$ with K_9 stated in (91), then lemma 3.6 (see (107)) implies $\sum_{i=1}^4 A_{i,\infty}^2\mu_i^2 \leq K_9 A_\infty^2 B_\infty^2\left(\mu_1+\mu_2+\mu_1\mu_2-\mu_3-\mu_4-\mu_3\mu_4\right)^2$ and we look for

$$K_{11} \geq K_9\sqrt{M_{14}M_{23}}\left(1+\max_{i=1,2,3,4}\left\{\frac{2\sqrt{M_{1234}}}{\sqrt{\overline{A_i^2}}}\right\}\right) + \max_{i=1,2,3,4}\left\{\frac{2A_{i,\infty}}{\sqrt{\overline{A_i^2}}}\right\}. \qquad (126)$$

We now suppose that $\overline{A_i^2} \leq \varepsilon_i^2$, where ε_i are constants to be specified later. In particular for $\overline{A} \leq \sqrt{\overline{A^2}} \leq \varepsilon_1$, we estimate (using $\overline{B} < \sqrt{M_{1234}}$, $\overline{C} \leq \sqrt{M_{13}}$, $\overline{D} \leq \sqrt{M_{14}}$ and $\overline{C}^2\,\overline{D}^2 = (\overline{C^2}-\overline{\delta_3^2})(\overline{D^2}-\overline{\delta_4^2})$ with (26) and (27) for the product $\overline{C^2}\,\overline{D^2}$) that

$$\begin{aligned}
&\|\overline{A}\,\overline{B} - \overline{C}\,\overline{D}\|_2^2 = \overline{A}^2\,\overline{B}^2 - 2\overline{A}\,\overline{B}\,\overline{C}\,\overline{D} + \overline{C}^2\,\overline{D}^2 \qquad (127)\\
&\geq -2\varepsilon_1\sqrt{M_{1234}\,M_{13}\,M_{14}} + (M_{13}-\varepsilon_1^2)(M_{14}-\varepsilon_1^2) - \overline{D^2}\,\overline{\delta_3^2} - \overline{C}^2\,\overline{\delta_4^2}.
\end{aligned}$$

Moreover by (26)–(29), a straightforward expansion yields

$$\|A-A_\infty\|_2^2 + \|B-B_\infty\|_2^2 + \|C-C_\infty\|_2^2 + \|D-D_\infty\|_2^2 \leq 2M_{1234}. \qquad (128)$$

Thus, combining the left hand side (111) with (128) and the right hand side with (122), (123), and (127) where $\overline{D^2}, \overline{C}^2 \le M_{1234}$, we must prove that

$$\begin{aligned} 2M_{1234} \quad &\le \quad K_{10}\left(M_{13}\, M_{14} - 2\varepsilon_1\sqrt{M_{1234}\, M_{13}\, M_{14}} - \varepsilon_1^2(M_{13}+M_{14}) + \varepsilon_1^4\right) \\ &\quad +(K_{11} - K_{10}\sqrt{M_{14}\, M_{23}} - K_{10}M_{1234})\left(\overline{\delta_1^2} + \overline{\delta_2^2} + \overline{\delta_3^2} + \overline{\delta_4^2}\right) \quad (129) \end{aligned}$$

We treat the first bracket on the right hand side of (129). After neglecting ε_1^4, we denote the non-negative solution of the (in terms of ε) quadratic equation $xy - 2\varepsilon\sqrt{M_{1234}\, xy} - \varepsilon^2(x+y) = h$ for $0 \le h \le xy$ by

$$\varepsilon(x,y,h) := -\frac{\sqrt{M_{1234}\, xy}}{x+y} + \sqrt{\frac{M_{1234}\, xy}{(x+y)^2} + \frac{xy-h}{x+y}}\,. \qquad (130)$$

In the present case, where $x = M_{13}$ and $y = M_{14}$, choosing in particular $h = \frac{xy}{2}$ confirms (129) with

$$\left.\begin{array}{l} K_{10} \ge \frac{4M_{1234}}{M_{13}M_{14}}\,, \\ K_{11} \ge K_{10}\left(\sqrt{M_{14}M_{23}} + M_{1234}\right), \end{array}\right\} \text{ for } \sqrt{\overline{A^2}} \le \varepsilon_1 := \varepsilon(M_{13}, M_{14}, \frac{M_{13}\, M_{14}}{2})\,. \qquad (131)$$

Similarly, for the cases $\overline{A_i^2} \le \varepsilon_i^2$, $i = 2,3,4$, we obtain the same K_{11} and

$$\begin{aligned} K_{10} \quad &\ge \quad \frac{4M_{1234}}{M_{23}M_{24}}\,, \quad \text{for } \sqrt{\overline{B^2}} \le \varepsilon_2 := \varepsilon(M_{23}, M_{24}, \frac{M_{23}\, M_{24}}{2})\,, \qquad (132) \\ K_{10} \quad &\ge \quad \frac{4M_{1234}}{M_{1i}M_{2i}}\,, \quad \text{for } \sqrt{\overline{A_i^2}} \le \varepsilon_i := \varepsilon(M_{1i}, M_{2i}, \frac{M_{1i}\, M_{2i}}{2})\,, \quad i = 3,4\,, \end{aligned}$$

and this yields (112) and (113). □

We are now able to state the entropy/entropy-dissipation estimate for the model (19) – (22), which holds for admissible functions regardless if or if not they are solutions.

Lemma 3.10 *Let a, b, c, d be (measurable) functions from Ω to $\mathbb{R}$ such that $0 \le a_i \le \|a_i\|_{L^\infty([0,t]\times[0,1])}$, and $\int_\Omega a_1 + a_3 = M_{13}$, $\int_\Omega a_1 + a_4 = M_{14}$, $\int_\Omega a_2 + a_3 = M_{23}$. Then,*

$$D(a_i) \ge \frac{4}{K_{12}(t)} \min\left\{\frac{1}{K_{10}}, \frac{\min\{d_a, d_b, d_c, d_d\}}{K_{11}P}\right\}(E(a_i) - E(a_{i,\infty})) \qquad (133)$$

where $K_{10}(M_{13}, M_{14}, M_{23})$ is defined in (112), $K_{11}(M_{13}, M_{14}, M_{23})$ in (113),

$$K_{12}(\|a_i\|_{L^\infty([0,t]\times[0,1])}, M_{13}, M_{14}, M_{23}) = \max_{i=1,2,3,4}\left\{\Phi(\|a_i\|_{L^\infty([0,t]\times[0,1])}, a_{i,\infty})\right\}\,, \qquad (134)$$

and $a_{i,\infty}$ for $i = 1,2,3,4$ are defined in (33).

Proof of lemma 3.10: Let square roots be denoted by capital letters, i.e. $A = \sqrt{a}, B = \sqrt{b},\ C = \sqrt{c},\ D = \sqrt{d}$. Using Poincaré's inequality and $(ab-cd)(\ln(ab)-\ln(cd)) \geq 4(AB-CD)^2$, we obtain the estimate

$$\begin{aligned} D(a,b,c,d) \ &\geq \ 4\left\|AB-CD\right\|_2^2 + \frac{4d_a}{P}\left\|A-\overline{A}\right\|_2^2 + \frac{4d_b}{P}\left\|B-\overline{B}\right\|_2^2 \\ &\quad + \frac{4d_c}{P}\left\|C-\overline{C}\right\|_2^2 + \frac{4d_d}{P}\left\|D-\overline{D}\right\|_2^2 . \end{aligned} \tag{135}$$

We show in the sequel that the r.h.s. of (135) is bounded below by the relative entropy $E(a,b,c,d) - E(a_\infty,b_\infty,c_\infty,d_\infty)$.

First, we use the conservation laws (27) and (28) to rewrite the relative entropy as

$$\begin{aligned} E(a,b,c,d) - E(a_\infty,b_\infty,c_\infty,d_\infty) &= \int_\Omega \Bigg(a\ln\frac{a}{a_\infty} - (a-a_\infty) \\ +b\ln\frac{b}{b_\infty} - (b-b_\infty) + c\ln\frac{c}{c_\infty} &- (c-c_\infty) + d\ln\frac{d}{d_\infty} - (d-d_\infty)\Bigg)\, dx, \end{aligned}$$

and we use the boundedness the function Φ defined in (12) (see [DF, lemma 2.1]) to estimate

$$\begin{aligned} E(a,b,c,d) - E(a_\infty,b_\infty,c_\infty,d_\infty) \ &\leq \ K_{12}\left(\|A-A_\infty\|_2^2 + \|B-B_\infty\|_2^2\right. \\ &\qquad \left. + \|C-C_\infty\|_2^2 + \|D-D_\infty\|_2^2\right), \end{aligned} \tag{136}$$

with K_{12} as defined above in (134). The statement of lemma 3.10 follows now from lemma 3.9 by comparison with (135). □

We now state a Cziszar-Kullback type inequality.

Lemma 3.11 *For all (measurable) functions $a_i : \Omega \to \mathbb{R}$, $i = 1,2,3,4$ such that $0 \leq a_i$ and $\int_\Omega (a_1+a_3) = M_{13}$, $\int_\Omega (a_1+a_4) = M_{14}$, $\int_\Omega (a_2+a_3) = M_{23}$, and $\int_\Omega (a_2+a_4) = M_{24}$, we have the inequality*

$$2\sqrt{2}(E(a,b,c,d) - E(a_\infty,b_\infty,c_\infty,d_\infty)) \geq \sum_{i=1}^4 \frac{\|a_i - a_{i,\infty}\|_1^2}{M_i},$$

with M_i defined in (32).

Proof of lemma 3.11: We define $q(a_i) = a_i \ln a_i - a_i$ and rewrite

$$E(a,b,c,d) - E(a_\infty,b_\infty,c_\infty,d_\infty) = \textstyle\sum_{i=1}^4 \int_\Omega a_i \ln\frac{a_i}{\overline{a_i}}\, dx \tag{137}$$

$$+q(\overline{a}) - q(a_\infty) + q(\overline{b}) - q(b_\infty) + q(\overline{c}) - q(c_\infty) + q(\overline{d}) - q(d_\infty). \tag{138}$$

Using the conservations (26),(29), we define moreover

$$\begin{aligned}
Q_{13}(M_{13},\overline{a}) &= q(\overline{a}) + q(M_{13}-\overline{a}) = Q_{13}(M_{13},\overline{c}) && \text{for } \overline{a},\overline{c} \in [0, M_{13}]\,,\\
Q_{14}(M_{14},\overline{a}) &= q(\overline{a}) + q(M_{14}-\overline{a}) = Q_{14}(M_{14},\overline{d}) && \text{for } \overline{a},\overline{d} \in [0, M_{14}]\,,\\
Q_{24}(M_{24},\overline{b}) &= q(\overline{b}) + q(M_{24}-\overline{b}) = Q_{24}(M_{24},\overline{d}) && \text{for } \overline{b},\overline{d} \in [0, M_{24}]\,,\\
Q_{23}(M_{23},\overline{b}) &= q(\overline{b}) + q(M_{23}-\overline{b}) = Q_{23}(M_{23},\overline{c}) && \text{for } \overline{b},\overline{c} \in [0, M_{23}]\,,
\end{aligned}$$

and may rewrite the line (138) in the following ways

$$\begin{aligned}
(138) &= Q_{13}(M_{13},\overline{a}) - Q_{13}(M_{13},a_\infty) + Q_{24}(M_{24},\overline{b}) - Q_{24}(M_{24},b_\infty)\\
&= Q_{14}(M_{14},\overline{a}) - Q_{14}(M_{14},a_\infty) + Q_{23}(M_{23},\overline{b}) - Q_{23}(M_{23},b_\infty)\\
&= Q_{13}(M_{13},\overline{c}) - Q_{13}(M_{13},c_\infty) + Q_{24}(M_{24},\overline{d}) - Q_{24}(M_{24},d_\infty)\\
&= Q_{23}(M_{23},\overline{c}) - Q_{23}(M_{23},c_\infty) + Q_{14}(M_{14},\overline{d}) - Q_{14}(M_{14},d_\infty)\,.
\end{aligned}$$

Since

$$\begin{aligned}
Q'_{13}(M_{13},\overline{a}) + Q'_{24}(M_{24},\overline{b}) = Q'_{13}(M_{13},\overline{c}) + Q'_{24}(M_{24},\overline{d}) = \ln \tfrac{a_\infty b_\infty}{c_\infty d_\infty} = 0\,,\\
Q'_{14}(M_{14},\overline{a}) + Q'_{23}(M_{23},\overline{b}) = Q'_{23}(M_{23},\overline{c}) + Q'_{14}(M_{14},\overline{d}) = \ln \tfrac{a_\infty b_\infty}{c_\infty d_\infty} = 0\,,
\end{aligned}$$

and

$$\begin{aligned}
Q''_{13}(M_{13},\overline{a}), Q''_{13}(M_{13},\overline{c}) \geq \tfrac{4}{M_{13}}\,, \quad Q''_{14}(M_{14},\overline{a}), Q''_{14}(M_{14},\overline{d}) \geq \tfrac{4}{M_{14}}\,,\\
Q''_{23}(M_{23},\overline{b}), Q''_{23}(M_{23},\overline{c}) \geq \tfrac{4}{M_{23}}\,, \quad Q''_{24}(M_{24},\overline{b}), Q''_{24}(M_{24},\overline{d}) \geq \tfrac{4}{M_{24}}\,,
\end{aligned}$$

we Taylor-expand (138) (using $\overline{a} - a_\infty = \overline{b} - b_\infty$ and $\overline{c} - c_\infty = \overline{d} - d_\infty$, respectively, so that the first order terms vanish) and estimate below by

$$(138) \geq \frac{|\overline{a}-a_\infty|^2}{M_1} + \frac{|\overline{b}-b_\infty|^2}{M_2} + \frac{|\overline{c}-c_\infty|^2}{M_3} + \frac{|\overline{d}-d_\infty|^2}{M_4}\,.$$

For the right hand side of (137), we estimate with Cziszar-Kullback-Pinsker inequality

$$\int_\Omega a_i \ln \frac{a_i}{\overline{a_i}}\, dx \geq \frac{1}{2\overline{a_i}} \|a_i - \overline{a_i}\|_1^2\,,$$

for which moreover $\overline{a_i} \leq M_i$, for $i = 1,2,3,4$ and obtain (by Young's inequality $\|a_i - a_{i,\infty}\|_1^2 \leq \sqrt{2}\|a_i - \overline{a_i}\|_1^2 + 2\sqrt{2}|\overline{a_i} - a_{i,\infty}|^2$)

$$E(a,b,c,d) - E(a_\infty,b_\infty,c_\infty,d_\infty) \geq \tfrac{\|a-a_\infty\|_1^2}{2\sqrt{2}M_1} + \tfrac{\|b-b_\infty\|_1^2}{2\sqrt{2}M_2} + \tfrac{\|c-c_\infty\|_1^2}{2\sqrt{2}M_3} + \tfrac{\|d-d_\infty\|_1^2}{2\sqrt{2}M_4}\,.$$

This ends the proof of lemma 3.11. □

Proof of proposition 1.2
By the entropy dissipation $\frac{d}{dt}E(a,b,c,d) = -D(a,b,c,d)$, lemma 3.10 yields

$$\frac{d}{dt}\ln(E(a_i) - E(a_{i,\infty})) \geq \frac{4}{K_{12}(t)} \min\left\{\frac{1}{K_{10}}, \frac{\min\{d_a, d_b, d_c, d_d\}}{K_{11}P}\right\}, \quad (139)$$

where $K_{12}(t) = \max_{i=1,2,3,4}\{\Phi(\|a_i\|_{L^\infty([0,t]\times[0,1])}, a_{i,\infty})\}$ with $\Phi(x,y)$ defined in (12) (this function is monotonely increasing in x). Moreover, it is easy to see that for $k > 1$,

$$\Phi(ky, y) = \frac{k\ln(k) - (k-1)}{(\sqrt{k}-1)^2} \leq \frac{\sqrt{k}+1}{\sqrt{k}-1}\ln(k), \qquad \forall k > 1. \quad (140)$$

Note that the factor $(\sqrt{k}+1)/(\sqrt{k}-1)$ is strictly monotone decreasing in k.

Next, we have by lemma 3.5 that $\|a_i\|_{L^\infty([0,t]\times[0,1])} < K_{19,i}(1 + t^{\frac{21}{2}})$, where $K_{19,i} \geq \sum_{i=1}^4 K_{17,i} \geq 2\sum_{i=1}^4 M_i^2$ may be small with M_i defined in (32). Hence, in order to apply (140), we estimate e.g. $\|a_i\|_{L^\infty([0,t]\times[0,1])} < \max\{K_{19,i}, 2a_{i,\infty}\}(1 + t^{\frac{21}{2}})$ and

$$\begin{aligned}\Phi(\|a_i\|_{L^\infty([0,t]\times[0,1])}, a_{i,\infty}) &\leq \Phi\left(\max\left\{\frac{K_{19,i}}{a_{i,\infty}}, 2\right\}\left(1 + t^{\frac{21}{2}}\right)a_{i,\infty}, a_{i,\infty}\right) \\ &\leq \frac{\sqrt{2}+1}{\sqrt{2}-1}\left(\ln\left(\max\left\{\frac{K_{19,i}}{a_{i,\infty}}, 2\right\}\right) + \ln\left(1 + t^{\frac{21}{2}}\right)\right)\end{aligned}$$

since $k \geq 2$ in (140) and therefore

$$K_{12}(t) \leq (\sqrt{2}+1)^2\left(\max_{i=1,2,3,4}\left\{\ln\left(\frac{K_{19,i}}{a_{i,\infty}}\right), \ln 2\right\} + \ln\left(1 + t^{\frac{21}{2}}\right)\right).$$

Next, integrating $\int_0^T (139)\,dt$ on the right hand side, we estimate

$$\int_0^T \frac{1}{\max\{\ln\frac{K_{19,i}}{a_{i,\infty}}, \ln 2\} + \ln(1 + t^{\frac{21}{2}})}dt \geq \frac{1}{(\max\{\frac{K_{19,i}}{a_{i,\infty}}, \ln 2\} + \frac{21}{2})}\frac{T}{\ln(e+T)}, \quad (141)$$

since both sides vanish at $T = 0$ and the time-derivatives of the left hand side can be estimated below by

$$\begin{aligned}&\geq \frac{1}{\max\{\ln\frac{K_{19,i}}{a_{i,\infty}}, \ln 2\} + \frac{21}{2}\ln(e+T)} \geq \frac{1}{(\max\{\frac{K_{19,i}}{a_{i,\infty}}, \ln 2\} + \frac{21}{2})}\frac{1}{\ln(e+T)} \\ &> \frac{1}{(\max\{\frac{K_{19,i}}{a_{i,\infty}}, \ln 2\} + \frac{21}{2})}\frac{1}{\ln(e+T)}\left(1 - \frac{T}{e+T}\frac{1}{\ln(e+T)}\right),\end{aligned}$$

which is the time-derivative of the right hand side of (141). Now, the statement of theorem 1.2 follows from integrating $\int_0^T (139)\, dt$ and the Cziszar-Kullback type lemma (3.11). ■

Proof of theorem 1.3

To establish an H^1-bound on the solution, lemma 3.5 and (63) yield

$$T \inf_{t\in[0,T]} \|\nabla_x a_i\|^2_{L^2(\Omega)} \leq \|\nabla_x a_i\|^2_{L^2([0,T]\times\Omega)} < 4K_{15,i}\, K_{19,i} \left(1 + T^{\frac{21}{2}}\right) ,$$

for all $T > 0$. Since the function $(T^{-1} + T^{\frac{19}{2}})$ assumes its minimum value at time $T = (2/19)^{2/21}$, there exists a time $\tau \in [0, (2/19)^{2/21}]$ when

$$\|\nabla_x a_i(\tau)\|^2_{L^2(\Omega)} \leq 4K_{15,i}\, K_{19,i}\, \frac{21}{2} \left(\frac{2}{19}\right)^{\frac{19}{21}} .$$

Next, multiplying the equations (19)–(22) formally with $\triangle_x a_i$ yields with Young's inequality

$$\frac{d}{dt}\int_\Omega |\nabla_x a_i|^2\, dx + d_i \int_\Omega (\triangle_x a_i)^2\, dx \leq \frac{1}{4d_i}\|ab - cd\|^2_{L^2(\Omega)} + d_i \int_\Omega (\triangle_x a_i)^2\, dx .$$

We integrate over a time interval $T > (2/19)^{2/21} \geq \tau$ this formula and obtain

$$\|\nabla_x a_i(T)\|^2_{L^2(\Omega)} \leq \|\nabla_x a_i(\tau)\|^2_{L^2(\Omega)} + \frac{1}{4d_i}\|ab - cd\|^2_{L^2([0,T]\times\Omega)} .$$

Using the bound (89), i.e. $\|ab - cd\|_{L^2([0,T]\times[0,1])} \leq \sum_{i=1}^4 K^2_{21,i}\,(1 + T^{\frac{19}{2}})$ with $K_{21,i}$ given in (88) we obtain

$$\|\nabla_x a_i(T)\|^2_{L^2(\Omega)} < K_{22}\left(1 + T^{19}\right) \quad \text{for} \quad T > (2/19)^{2/21} , \tag{142}$$

with the constant

$$K_{22} = 4K_{15,i}\, K_{19,i}\frac{21}{2}\left(\frac{2}{19}\right)^{\frac{19}{21}} + \frac{1}{2d_i}\left(\sum_{i=1}^4 K^2_{21,i}\right)^2 . \tag{143}$$

This formal argument is made rigorous by approximating the solution (see e.g. [MP]).

Next, see e.g. [Tay] for the Gagliardo-Nirenberg-Moser interpolation inequality

$$\|a_i\|_{L^\infty(\Omega)} \leq G(\Omega)\|\nabla_x a_i\|^{\frac{1}{2}}_{L^2(\Omega)}\|a_i\|^{\frac{1}{2}}_{L^2(\Omega)} , \tag{144}$$

with a constant $G(\Omega)$ on a smoothly bounded Ω. Then, interpolating the almost exponentially decaying L^1-norm of theorem 1.2 for $T > (2/19)^{2/21} \geq \tau$, we get

$$\|a_i(T)\|_{L^\infty(\Omega)} \leq a_{i,\infty} + \|a_i - a_{i,\infty}\|_{L^\infty(\Omega)}$$
$$\leq a_{i,\infty} + G(\Omega)\|\nabla_x (a_i - a_{i,\infty})\|_{L^2}^{\frac{1}{2}} \|a_i - a_{i,\infty}\|_{L^\infty}^{\frac{1}{4}} \|a_i - a_{i,\infty}\|_{L^1}^{\frac{1}{4}} \leq K_{13,i}\,,$$

where (142), lemma 3.5 and theorem 1.2 lead to the constants $K_{13,i}$ given in (36). Moreover, the value of $K_{13,i}$ for $0 < \tau \leq (2/19)^{2/21}$ follows from lemma 3.5. Finally, using this global L^∞-bound, the right hand side of (139) is bounded below by a constant and the exponential decay stated in the theorem can be obtained. ■

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