Control and stabilization of Schrödinger equations with bilinear control
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Contents

Chapter 1

Introduction

1.1 The studied system, goal

Let us consider a quantum particle in a 1D infinite square potential well. It is represented by a wave function

$$\begin{array}{cccccc} \psi : & \mathbb{R} & \times & (0,1) & \rightarrow & \mathbb{C} \\ & (t & , & x) & \mapsto & \psi(t,x) \end{array}$$

where x is the space variable and $(0,1)$ is the bottom of the well. The physical meaning of $|\psi(t,x)|^2 dx$ is the probability of the particle to be in the volume dx surrounding the point x at time t. Thus, the wave function $\psi(t,.)$ lives on the $L^2((0,1),\mathbb{C})$-sphere,

$$\int_0^1 |\psi(t,x)|^2 dx = 1, \forall t \in \mathbb{R}.$$

We study a quantum particle subjected to an electric field, with amplitude $u : t \in [0,T] \rightarrow u(t) \in \mathbb{R}$. Then, the time evolution of the wave function is given by the Schrödinger equation,

$$\left\{ \begin{array}{l} i\frac{\partial \psi}{\partial t}(t,x) = -\frac{\partial^2 \psi}{\partial x^2}(t,x) - u(t)\mu(x)\psi(t,x), t \in (0,T), x \in (0,1), \\ \psi(t,0) = \psi(t,1) = 0, \end{array} \right. \tag{1.1.1}$$

where μ is the dipolar moment. The goal of this course is to propose electric fields u, moving the solution of (1.1.1) from a given initial state ψ_0, to a desired final state ψ_f. The system (1.1.1) is a bilinear control system in which

- the state is the wave function ψ, with $\|\psi(t,.)\|_{L^2(0,1)} = 1, \forall t \in [0,T]$,
- the control is the real valued function u.

Let us introduce the operator

$$D(A) := H^2 \cap H_0^1((0,1),\mathbb{C}), A\varphi := -\varphi'' \tag{1.1.2}$$

and its eigenfunctions and eigenvalues.

$$\varphi_k(x) := \sqrt{2}\sin(k\pi x), \lambda_k := (k\pi)^2, \forall k \in \mathbb{N}^*. \tag{1.1.3}$$

Then, $(\varphi_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(0,1)$ and

$$\psi_k(t,x) := \varphi_k(x)e^{-i\lambda_k t}, \forall k \in \mathbb{N}^* \tag{1.1.4}$$

is a solution of (1.1.1) with $u \equiv 0$, called eigenstate, or ground state, when $k = 1$. Typically, for a given initial state ψ_0, we search controls u such that the solution of (1.1.1) with $\psi(0,x) = \psi_0(x)$, reaches the ground state, either in finite time T, i.e. $\psi(T) = \psi_1(\tau)$ for some $T, \tau \in \mathbb{R}$, or in infinite time (stabilisation), i.e. $\lim_{t\to+\infty} \|\psi(t) - \psi_1(t+\tau)\|_{L^2} = 0$.

The study of the controllability properties of the system (1.1.1) is rather difficult because it is an *infinite dimensional* system (i.e. the state ψ lives in an infinite dimensional space, which is, for example $L^2((0,1),\mathbb{C})$). For pedagogical purposes, in this course, we will first work on the following *finite dimensional* system, which is very similar to (1.1.1) and corresponds to its Galerkin approximation,

$$i\frac{dX}{dt}(t) = H_0X(t) - u(t)H_1X(t), t \in (0,T), \tag{1.1.5}$$

where $N \in \mathbb{N}^*$, $X : t \in [0,T] \mapsto X(t) \in \mathbb{C}^N$, H_1 is a $N \times N$ symmetric matrix with real valued coefficients,

$$H_0 = \begin{pmatrix} \lambda_1 & 0 & 0 & ... & 0 \\ 0 & \lambda_2 & 0 & ... & 0 \\ 0 & 0 & \lambda_3 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \lambda_N \end{pmatrix} \text{ and } 0 \leqslant \lambda_1 < \lambda_2 < \lambda_3 < ... < \lambda_N. \tag{1.1.6}$$

This is a bilinear control system in which

- the state is $t \in [0,T] \mapsto X(t) \in \mathbb{C}^N$ with $\|X(t)\| = 1, \forall t \in [0,T]$,
- the control is $t \in [0,T] \mapsto u(t) \in \mathbb{R}$.

Let us introduce eigenvectors of H_0, $\varphi_1 := (1,0,0,0...,0)$, $\varphi_2 := (0,1,0,0...,0)$, $\varphi_3 := (0,0,1,0...,0)$,...,$\varphi_N := (0,0,0,0...,1)$,

$$H_0\varphi_k = \lambda_k\varphi_k, \forall k \in \{1,...,N\}.$$

Then

$$\psi_k(t) := \varphi_k e^{-i\lambda_k t}, \forall t \in \mathbb{R}, \forall k \in \{1, ..., N\}$$

is a solution of (1.1.5) associated to $u \equiv 0$, called eigenstate, or ground state when $k = 1$.

The same notations $\psi_k, \varphi_k, \lambda_k$ are used, for the finite dimensional system and the infinite dimensional system. No confusion is possible: within any chapter of this course, we consider either only the finite dimensional model, or only the infinite dimensional one.

1.2 Iterated Lie brakets in finite dimension

The Lie brackets are a powerful tool to study the controllability of finite dimensional systems (see [25] for more details thank in this section).

Definition 1 *Let $n \in \mathbb{N}^*$ and $A, B \in \mathcal{M}_n(\mathbb{C})$. The **Lie bracket** of A and B is the $n \times n$ matrix $[A, B] := AB - BA$.*

The Lie brackets appear naturally in the study of the controllability because they characterise directions in which it is possible to move. Let us explain it on our example. We consider the ODE (1.1.5) with the initial condition $X(0) = \varphi_1$ and we assume $\lambda_1 = 0$.

One know how to move in the direction of $\pm H_1\varphi_1$. Indeed, let $\eta \in \mathbb{R}$, $\epsilon \in (0, 1)$ and let us consider the control function on $[0, \epsilon]$ defined by

$$u(t) := \eta, \forall t \in [0, \epsilon].$$

Then, we have

$$\begin{aligned} X(\epsilon) &= \exp[(H_0 - \eta H_1)\epsilon]\varphi_1 \\ &= [I + (H_0 - \eta H_1)\epsilon + o(\epsilon)]\varphi_1 \\ &= \varphi_1 - \epsilon\eta H_1\varphi_1 + o(\epsilon) \end{aligned}$$

when $\epsilon \to 0$, because $H_0\varphi_1 = \lambda_1\varphi_1 = 0$. Thus, we can move in the direction of $\pm H_1\varphi_1$.

Let us explain how to move in the direction of $\pm[H_0, H_1]\varphi_1$. For $\eta \in \mathbb{R}$, $\epsilon \in (0, 1)$, we consider the control function defined on $[0, 2\epsilon]$ by

$$\begin{aligned} &u(t) := -\eta \text{ for } t \in (0, \epsilon), \\ &u(t) := \eta \text{ for } t \in (\epsilon, 2\epsilon). \end{aligned}$$

Then, we have

$$\begin{aligned} & X(2\epsilon) = \exp[(H_0 - \eta H_1)\epsilon] \exp[(H_0 + \eta H_1)\epsilon]\varphi_1 \\ = & \Big(I + (H_0 - \eta H_1)\epsilon + \tfrac{1}{2}(H_0 - \eta H_1)^2\epsilon^2 + o(\epsilon^2)\Big) \\ & \times\Big(I + (H_0 + \eta H_1)\epsilon + \tfrac{1}{2}(H_0 + \eta H_1)^2\epsilon^2 + o(\epsilon^2)\Big)\varphi_1 \\ = & \Big(I + 2\epsilon H_0 + \epsilon^2\{2H_0^2 + \eta[H_0, H_1]\} + o(\epsilon^2)\Big)\varphi_1 \\ = & \varphi_1 + \epsilon^2\eta[H_0, H_1]\varphi_1 + o(\epsilon^2) \end{aligned}$$

when $\epsilon \to 0$. Thus, we can move in the direction of $\pm[H_0, H_1]\varphi_1$.

Now, let us explain how to move in the direction of $\pm[H_0, [H_0, H_1]]$. Notice that

$$[H_0, [H_0, H_1]]\varphi_1 = \Big(H_0(H_0H_1 - H_1H_0) - (H_0H_1 - H_1H_0)H_0\Big)\varphi_1 = H_0^2H_1\varphi_1.$$

Let $\eta \in \mathbb{R}$, $\epsilon \in (0, 1)$, and $\phi \in C^2([0, 1], \mathbb{R})$ be such that

$$\phi(t) = \phi'(t) = 0 \text{ at } t = 0, 1, \tag{1.2.1}$$

and $\int_0^1 \phi \neq 0$. We consider the control function defined on $[0, \eta]$ by

$$u(t) := \epsilon\phi''\left(\frac{t}{\eta}\right), \forall t \in [0, \eta].$$

Let Y be the solution of the linearised system around $(\psi \equiv \varphi_1, u \equiv 0)$,

$$\begin{cases} i\frac{dY}{dt} = H_0Y - u(t)H_1\varphi_1, \\ Y(0) = 0. \end{cases}$$

Using two integrations by parts in which the boundary terms vanish because of (1.2.1), we get

$$\begin{aligned} Y(\eta) & = i\int_0^\eta \epsilon\phi''\left(\tfrac{t}{\eta}\right) e^{-iH_0(\eta-t)}H_1\varphi_1 dt \\ & = -i\epsilon\eta^2\int_0^\eta \phi\left(\tfrac{t}{\eta}\right) H_0^2 e^{-iH_0(\eta-t)}H_1\varphi_1 dt \\ & = -i\epsilon\eta^3\left(\int_0^1 \phi\right) H_0^2H_1\varphi_1 + O(\epsilon\eta^4). \end{aligned} \tag{1.2.2}$$

because

$$\|e^{-iH_0(\eta-t)} - 1\| \leqslant C\eta, \forall t \in [0, \eta].$$

Let us prove that

$$X(\eta) - \varphi_1 - Y(\eta) = O(\epsilon^2\eta^2). \tag{1.2.3}$$

Indeed, we have

$$i\frac{d(X - \varphi_1 - Y)}{dt} = [H_0 - uH_1](X - \varphi_1 - Y) + uH_1Y,$$

thus

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|X-\varphi_1-Y\|^2 &= \Re\langle X-\varphi_1-Y, -i[H_0-uH_1](X-\varphi_1-Y)-iuH_1Y\rangle \\ &= -u\Im\langle X-\varphi_1-Y, H_1Y\rangle \\ &\leqslant |u|\|X-\varphi_1-Y\|\|H_1\|\|Y\|, \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\|X-\varphi_1-Y\| &= \frac{1}{2\|X-\varphi_1-Y\|}\frac{d}{dt}\|X-\varphi_1-Y\|^2 \\ &\leqslant |u|\|H_1\|\|Y\|. \end{aligned}$$

Integrating this inequality on $(0,\eta)$, we get

$$\|(X-\varphi_1-Y)(\eta)\| \leqslant \|H_1\|\|u\|_{L^1(0,\eta)}\|Y\|_{L^\infty(0,\eta)}.$$

Moreover, we have

$$\|u\|_{L^1(0,\eta)} = \int_0^\eta \epsilon\left|\phi''\left(\frac{t}{\eta}\right)\right| dt = \epsilon\eta\int_0^1 |\phi''|,$$

$$\begin{aligned} \|Y(t)\| &= \left\| i\int_0^t \epsilon\phi''\left(\frac{t}{\eta}\right) e^{-iH_0(t-s)}H_1\varphi_1 ds\right\| \\ &\leqslant \epsilon\eta\left(\int_0^1|\phi''|\right)\|H_1\|. \end{aligned}$$

Thus,

$$\|(X-\varphi_1-Y)(T)\| \leqslant C\epsilon^2\eta^2,$$

which proves (1.2.3). From (1.2.2) and (1.2.3), we get

$$X(\eta) = -i\epsilon\eta^3\left(\int_0^1\phi\right)H_0^2H_1\varphi_1 + O(\epsilon\eta^4+\epsilon^2\eta^2).$$

Note that taking $\eta := \epsilon^{1/6}$, we get

$$X(\eta) = -i\epsilon^{3/2}\left(\int_0^1\phi\right)[H_0,[H_0,H_1]]\varphi_1 + O(\epsilon^{5/3}).$$

Thus, we can move in the direction of $\pm[H_0,[H_0,H_1]]\varphi_1$.

Adapting the previous arguments, one may prove that the control

$$u(t) := \epsilon\phi^k\left(\frac{t}{\eta}\right), \forall t\in[0,\eta],$$

where $\phi(t) = \phi'(t) = ... = \phi^{(k-1)}(t) = 0$ at $t = 0, 1$ allows to move in the direction of $ad_{H_0}^k(H_1)$, where

$$ad_{H_0}(f) := [H_0, f].$$

Using the Lie brackets, one may prove the following result, due to Albertini and D'Alessandro in [4].

Theorem 1 *The system (1.1.5) is globally controllable if and only if* $Lie(H_0, H_1)$ *is isomorphic (conjugated) to*

- $su(N)$ *if* N *is odd,*
- $su(N)$ *or* $sp(N/2)$ *if* N *is even.*

Let us also mention that the controllability of bilinear finite dimensional Schrodinger systems has been studied in (see for example [4], [5], [22], [43], [47], [49]).

We also refer to the following works for results about the controllability of finite dimensional quantum systems [6], [21], [26], [3], [51], [19], [50], [18], [20], [33]. The books [31], [1] can be useful for the study of these systems.

In conclusion to this subsection, the Lie brackets are a powerful tool to study the controllability of finite dimensional bilinear systems.

1.3 Iterated Lie brackets in infinite dimension

In this subsection, we explain why iterated Lie brackets are less powerful in infinite dimension than in finite dimension.

1.3.1 Sometimes useful

First, let us show, on an example, how iterated Lie brackets can sometimes still be useful for studying the controllability in infinite dimension. This example is borrowed from [44] and [42]. We consider the following system

$$i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} + x^2\psi - u(t)x\psi, x \in \mathbb{R}, t \in (0, T). \tag{1.3.1}$$

It is a control system in which

- the state is the wave function ψ with
$$\int_{\mathbb{R}} |\psi(t, x)|^2 dx = 1, \forall t \in [0, T],$$
- the control is the real valued function $u : [0, T] \to \mathbb{R}$, which corresponds to a classical electro-magnetic field.

The free Hamiltonian

$$H_0(\psi) := -\psi'' + x^2\psi$$

corresponds to the usual harmonic oscillator. With our previous notations, we define the vector fields

$$f_0(\psi) := -\psi'' + x^2\psi,$$

$$f_1(\psi) := x\psi.$$

At a formal level, we have

$$\begin{aligned}
&[f_0, f_1](\psi) = -2\psi',\\
&[f_0, [f_0, f_1]](\psi) = 4x\psi = 4f_1(\psi),\\
&[f_1, [f_0, f_1]](\psi) = 2\psi.
\end{aligned}$$

Thus, the Lie algebra generated by f_0 and f_1 is of dimension 4 : it is the linear space generated by f_0, f_1, $[f_0, f_1]$ and Id. Hence, one would expect that the dimension of the reachable set from a given point should be of dimension at most 4. Indeed, one can check, by direct computations, that the intuition given by the above arguments is indeed correct. We present an heuristic of these computations and we refer to [42] and [24, page 51] for precise arguments. Let $p, q : [0, T] \to \mathbb{R}$ be defined by

$$q(t) := \int_{\mathbb{R}} x|\psi(t, x)|^2 dx,$$

$$p(t) := -2\Im \int_{\mathbb{R}} \overline{\psi'}\psi.$$

Note that $q(t)$ is the average position and $p(t)$ is the average momentum of the quantum system. The dynamics of the couple (p, q) is given by

$$\left\{\begin{array}{l} \dot{q} = p,\\ \dot{p} = -4q + 2u. \end{array}\right.$$

By the Kalman rank condition (see, for example, [25, Theorem 1.16 page 9]) this linear control system is globally controllable. Following [24, page 51], let us define $\phi \in C^0([0, T], S)$ (S is the $L^2(\mathbb{R}, \mathbb{C})$-sphere) by

$$\phi(t, x) := \psi(t, x + q)e^{-i\frac{p}{2}x + ir}$$

where

$$r(t) := \int_0^t \Big(q(s)^2 - \frac{3}{4}p(s)^2 - u(s)q(s)\Big)ds.$$

Then, straightforward computations lead to

$$\frac{\partial \phi}{\partial t} = -\frac{\partial^2 \phi}{\partial x^2} + x^2\phi,$$

thus the evolution of ϕ does not depend on the control u. This allows to share the state ψ into two parts

- a 2 dimensional controllable part (p, q), that corresponds to the classical dynamics of the particle (average position q and average momentum p)

- an infinite dimensional non controllable part ϕ.

With this precise description of ψ, it is not hard to check that, given $\psi_0 \in S$, the reachable set

$$\Big\{\psi(T); T > 0, u : (0,T) \to \mathbb{R}, \psi \text{ solution of (1.3.1) such that } \psi(0) = \psi_0\Big\}$$

is contained in a submanifold of S of dimension 4.

1.3.2 Sometimes less powerful

Now, let us show, on an example, why iterated Lie brackets may sometimes be less powerful in infinite dimension than in finite dimension. This example is one of the systems studied in this document. We consider the following control system

$$i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} - u(t)x^2\psi, x \in (0,1), t \in (0,T). \tag{1.3.2}$$

It is a control system in which

- the state is the wave function ψ with
$$\int_0^1 |\psi(t,x)|^2 dx = 1, \forall t \in [0,T],$$
- the control is the function $u : [0,T] \to \mathbb{R}$, which corresponds to an electric field.

It is proved in [10] that this control system is locally controllable in $H^5((0,1),\mathbb{C})$, in any positive time $T > 0$, with controls in $H_0^1((0,T),\mathbb{R})$, around the ground state $\psi_1(t,x) := \sqrt{2}\sin(\pi x)e^{-i\pi^2 t}$. We define the operators f_0 and f_1

$$\begin{array}{ll} D(f_0) := H^2 \cap H_0^1((0,1),\mathbb{C}) & f_0(\psi) := -\psi'', \\ D(f_1) := L^2((0,1),\mathbb{C}) & f_1(\psi) := x^2\psi. \end{array}$$

Let us compute the iterated Lie bracket at the point $\varphi_1(x) := \sqrt{2}\sin(\pi x)$. Since $\varphi_1 \in D(f_0)$, we can compute

$$[f_0, f_1](\varphi_1) = -4x\varphi_1' - 2\varphi_1,$$
$$[f_1,[f_0,f_1]](\psi) = 8x^2\varphi_1 = 8f_1(\varphi_1).$$

Notice that $[f_0,f_1](\varphi_1)$ does not belong to $D(f_0)$ because $[f_0,f_1](\varphi_1)(1) = 4\sqrt{2}\pi \neq 0$. Thus, in order to give a sense to the Lie bracket $[f_0,[f_0,f_1]$, one needs to extend the definition of f_0 to functions that do not vanish at $x = 0, 1$. A natural choice is

$$f_0(\psi) := -\psi'' + \psi(0)\delta_0' - \psi(1)\delta_1' \tag{1.3.3}$$

because, with this choice, we have

$$\langle f_0(\psi), \widetilde{\psi} \rangle = \langle \psi, f_0(\widetilde{\psi}) \rangle, \forall \psi \in D(f_0), \forall \widetilde{\psi} \in H^2((0,1), \mathbb{C}),$$

in the sense

$$-\int_0^1 \psi''(x)\widetilde{\psi}(x)dx = -\int_0^1 \psi(x)\widetilde{\psi}''(x)dx - \psi'(1)\widetilde{\psi}(1) + \psi'(0)\widetilde{\psi}(0),$$

which is an integration by parts. With the definition (1.3.3), we get

$$[f_0, [f_0, f_1]](\psi) = -8 f_0(\psi) + 4\psi'(1)\delta_1'$$

But then, again, $[f_0, [f_0, [f_0, f_1]]]$ is not well defined. Moreover, even if we could give a sense to any iterated Lie bracket, because of the presence of Dirac masses, it would not be clear which space the Lie algebra should generate in case of local controllability. Therefore, the way the Lie algebra rank condition could be used in infinite dimension is not clear.

For applications of the iterated Lie bracket to get positive controllability results for partial differential equations, let us mention, for example, the case of Navier-Stokes and Euler equations by Agrachev and Sarychev [2] and by Shirikyan [46].

In conclusion, for infinite dimensional systems, there are cases where the iterated Lie brackets provide the right intuition. However, it is not always the case, thus, the Lie brackets are less powerful in infinite dimension than in finite dimension. As a consequence, the exact controllability of an infinite dimensional bilinear system (i.e. modelled by a partial differential equation) is a more difficult problem than the controllability of a finite dimensional bilinear system (i.e. modelled by an ordinary differential equation).

The goal of this course is to develop technics for the study of the controllability of (1.1.5), that do not rely on Lie brackets, in order to use them on the infinite dimensional model (1.1.1).

1.4 More bibliography

The local controllability and the controllability between eigenstates of systems similar to (1.1.1) is the subject of [11], [14], [13], [12], [10]. The spectral controllability of the linearised system around the ground state, of (1.1.1), in 2D and 3D (space dimension) is the subject of [16]. Lyapunov technics are presented in [15], [32].

Results on distributed and boundary exact controllability for **linear** Schrödinger equations are the subjects of [37], [35], [36], [39]. In these cases, the controllability is equivalent to the validity of an observability inequality which is proved

- thanks to the multiplier method in [39] under some geometric condition,
- thanks to Carlemann estimates in [35] and [36],
- thanks to microlocal analysis in [37] under the geometric control condition.

However, the geometric control condition is not necessary for the controllability. This was proved for the plate equation (an equation very similar to Schrödinger equation) by Haraux and Jaffard on a rectangle in [28] and [30], and by Burq on more general geometries in [23].

Concerning bilinear control systems of Schrödiger type, let us first mention the non controllability results of [48] and [29]. In [48], the PDE is linear and in [29], the PDE may be nonlinear. Let us also mention the positive controllability result [17].

Optimal control techniques have been investigated for Schrödinger equations with a non linearity of Hartee type in [27], [7] and [8]. An algorithm for the calculus of such optimal controls is studied in [9]. Other algorithms for the computation of controls are presented in [40] and [41].

1.5 Structure of this course

This course is organised as follows.

In chapter 2, we study the exact controllability, locally around the ground state, of the finite dimensional model (1.1.5). The proof is relies on the linearization principle. First, we prove that, under suitable assumptions on the matrices H_0 and H_1, the linearised system around the ground state is controllable. Then, we prove that the end-point map is C^1. Finally, we conclude by applying the inverse mapping theorem.

In chapter 3, we study the same property for the infinite dimensional model (1.1.1). The complete proof is too long to be presented in this course. Precisely, the regularity C^1 of the end-point map is quite difficult to prove. Thus, we only prove the controllability of the linearised system, thanks to the moment method. The necessary results from moment theory are proved in the next chapter.

In chapter 4, we study the solvability of an infinite trigonometric moment problem, in which the frequencies have an infinite asymptotic gap.

In chapter 5, we study the stabilisation of the ground state, for the finite dimensional model (1.1.5), with Lyapunov technics. In particular, we use the LaSalle invariance principle.

In chapter 6, we study the same property for the infinite dimensional model (1.1.1).

Chapter 2

Control in finite dimension

This chapter is dedicated to the local exact controllability of (1.1.5) around the ground state ψ_1.

Definition 2 *The system (1.1.5) is locally controllable in time T (with small controls) around the trajectory ($\psi = \psi_1, u \equiv 0$) if, for every $\epsilon > 0$, there exists $\delta > 0$ such that, for every $X_f \in \mathbb{C}^N$ with $\|X_f\| = 1$ and $\|X_f - \psi_1(T)\| < \delta$, there exists $u \in C^0([0,T],\mathbb{R})$ with $\|u\|_{L^\infty(0,T)} < \epsilon$ such that the solution of (1.1.5) with*

$$X(0) = \varphi_1, \tag{2.0.1}$$

satisfies $X(T) = X_f$.

2.1 Well posedness of the Cauchy-problem

Theorem 2 *Let $T > 0$, $u \in C^0(([0,T],\mathbb{R})$ and $X_0 \in \mathbb{C}^N$ with $\|X_0\| = 1$. There exists a unique solution $X \in C^1([0,T],\mathbb{C}^N)$ of (1.1.5) such that*

$$X(0) = X_0. \tag{2.1.1}$$

Moreover $\|X(t)\| = 1, \forall t \in [0,T]$. These solutions are continuous with respect to (X_0, u) in the following sense: for every $u^1, u^2 \in C^0([0,T],\mathbb{R})$, for every $X_0^1, X_0^2 \in \mathbb{C}^\mathbb{N}$ with $\|X_0^1\| = \|X_0^2\| = 1$, we have

$$\|X^1 - X^2\|_{L^\infty(0,T)} \leqslant \|X_0^1 - X_0^2\| + \|H_1\| \|u^1 - u^2\|_{L^1(0,T)}. \tag{2.1.2}$$

Proof of Theorem 2: The function $f(t,x) := -iH_0x + iu(t)H_1x$ is continuous with respect to $(t,x) \in [0,T] \times \mathbb{C}^N$ and linear in x, thus the Cauchy-Lipschitz theorem (in linear form) ensures the existence of a unique solution of (1.1.5), (2.1.1), defined on the whole interval $[0,T]$: $X \in C^1([0,T]),\mathbb{C}^N)$.

Then, the map $t \mapsto \|X(t)\|^2$ is C^1 over $[0,T]$ and for every $t \in [0,T]$, we have

$$\begin{aligned} \frac{d}{dt}\|X(t)\|^2 &= 2\Re\langle X(t), \frac{dX}{dt}(t)\rangle \\ &= 2\Re\langle X(t), -i[H_0 - u(t)H_1]X(t)\rangle \\ &= 0 \end{aligned}$$

because the matrix $H_0 - u(t)H_1$ is hermitian, $\forall t \in [0,T]$. Therefore $\|X(t)\| = 1, \forall t \in [0,T]$. From

$$\begin{cases} i\frac{d(X^1 - X^2)}{dt} = [H_0 - u_1(t)H_1](X^1 - X^2) - (u^1 - u^2)(t)H_1X_2, \\ (X^1 - X^2)(0) = X_0^1 - X_0^2, \end{cases}$$

we get

$$\begin{aligned} & \tfrac{1}{2}\tfrac{d}{dt}\|(X^1 - X^2)(t)\|^2 \\ = \;& \Re\langle (X^1 - X^2)(t), -i[H_0 - u^1(t)H_1](X^1 - X^2)(t) + i(u^1 - u^2)(t)H_1X^2(t)\rangle \\ = \;& (u^1 - u^2)(t)\Im\langle (X^1 - X^2)(t), H_1X^2(t)\rangle \\ \leqslant \;& |(u^1 - u^2)(t)|\|H_1\|\|(X^1 - X^2)(t)\|. \end{aligned}$$

Thus,

$$\frac{d}{dt}\|(X_1 - X_2)(t)\| = \frac{1}{2\|(X_1 - X_2)(t)\|}\frac{d}{dt}\|(X^1 - X^2)(t)\|^2 \leqslant \|H_1\||(u^1 - u^2)(t)\|.$$

Integrating this relation over $[0,t]$, we get

$$\|(X_1 - X_2)(t)\| \leqslant \|X_0^1 - X_0^2\| + \|H_1\|\int_0^t |(u^1 - u^2)(s)\|ds, \forall t \in [0,T],$$

which gives (2.1.2). □

2.2 Local controllability: the non pathological case

Theorem 3 *Let $T > 0$. We assume*

$$\langle H_1\varphi_1, \varphi_k\rangle \neq 0, \forall k \in \{1, ..., N\}. \tag{2.2.1}$$

There exists $\delta > 0$ and a C^1-map

$$\Gamma : \mathcal{V}_T \to C^0([0,T], \mathbb{R})$$

where

$$\mathcal{V}_T := \{X \in \mathbb{C}^N; \|X\| = 1 \text{ and } \|X - \psi_1(T)\| < \delta.\}$$

such that $\Gamma[\psi_1(T)] = 0$ and, for every $X_f \in \mathcal{V}_T$, the solution of (1.1.5), (2.0.1) with $u = \Gamma(X_f)$ satisfies $X(T) = X_f$.

This theorem is a consequence of the inverse mapping theorem.

Theorem 4 *Let E_1, E_2 be Banach spaces, U be an open neighbourhood of 0 in E_1, and $F : U \to E_2$ be a C^1-map. We assume that $dF(0) : E_1 \to E_2$ has a continuous right inverse $dF(0)^{-1} : E_2 \to E_1$ (i.e. $dF(0)[dF(0)^{-1}.y] = y, \forall y \in E_2$). Then, there exists an open neighbourhood V of $F(0)$ in E_2 and a C^1-map $G : V \to U$ such that $G(F(0)) = 0$ and $F(G(y)) = y, \forall y \in V$.*

In order to apply Theorem 4, we will need the following result.

Proposition 1 *Let $T > 0$ and*

$$\begin{array}{rccc} \Theta: & C^0([0,T],\mathbb{R}) & \to & C^0([0,T],\mathbb{C}^N) \\ & u & \mapsto & X \end{array}$$

where X solves (1.1.5), (2.0.1). Then, Θ is C^1 and, for every $u \in C^0([0,T],\mathbb{R})$, $d\Theta(u).v = Y$ where

$$\left\{ \begin{array}{l} i\frac{dY}{dt}(t) = [H_0 - u(t)H_1]Y(t) - v(t)H_1X(t), \\ Y(0) = 0 \end{array} \right. \tag{2.2.2}$$

and $X = \Theta(u)$.

Proof of Proposition 1: Let $u \in C^0([0,T],\mathbb{R})$. The linear map

$$\begin{array}{ccc} C^0([0,T],\mathbb{R}) & \to & C^0([0,T],\mathbb{C}^N) \\ v & \mapsto & Y \end{array}$$

where Y solves (2.2.2) is continuous. Indeed, we have

$$\begin{array}{ll} \frac{1}{2}\frac{d}{dt}\|Y(t)\|^2 & = \Re\langle Y(t), -i[H_0 - u(t)H_1]Y(t) + iv(t)H_1X(t)\rangle \\ & = v(t)\Im\langle Y(t), H_1X(t)\rangle \\ & \leqslant |v(t)|\|H_1\|\|Y(t)\|. \end{array}$$

Thus,

$$\frac{d}{dt}\|Y(t)\| = \frac{1}{2\|Y(t)\|}\frac{d}{dt}\|Y(t)\|^2 \leqslant |v(t)|\|H_1\|.$$

Integrating this relation over $(0,t)$, for any $t \in [0,T]$, we get

$$\|Y\|_{L^\infty(0,T)} \leqslant \|H_1\|\|v\|_{L^1(0,T)} \leqslant T\|H_1\|\|v\|_{L^\infty(0,T)}. \tag{2.2.3}$$

Let $u, v \in C^0([0,T],\mathbb{R})$ and $X, Y, \tilde{X}$ be the solutions of (1.1.5)-(2.0.1), (2.2.2), and

$$\left\{ \begin{array}{l} i\frac{d\tilde{X}}{dt} = [H_0 - (u+v)H_1]\tilde{X}, \\ \tilde{X}(0) = \varphi_1. \end{array} \right.$$

Let us prove that

$$\|X - \varphi_1 - Y\|_{L^\infty(0,T)} = o(\|v\|_{L^\infty(0,T)}). \tag{2.2.4}$$

From

$$\begin{cases} i\frac{d(X-\varphi_1-Y)}{dt} = [H_0-(u+v)H_1](X-\varphi_1-Y) - vH_1Y, \\ (X-\varphi_1-Y)(0)=0, \end{cases}$$

we get,

$$\|X-\varphi_1-Y\|_{L^\infty(0,T)} \leqslant \|H_1\|\|v\|_{L^1(0,T)}\|Y\|_{L^\infty(0,T)}.$$

Thanks to (2.2.3), we deduce that

$$\|X-\varphi_1-Y\|_{L^\infty(0,T)} \leqslant T^2\|H_1\|^2\|v\|^2_{L^\infty(0,T)},$$

which proves (2.2.4). □

Proposition 2 *We assume (2.2.1). Let $T>0$. There exists a continuous map $\Gamma_L : T_S\psi_1(T) \rightarrow C^0([0,T],\mathbb{R})$ such that, for every $Y_f \in T_S\psi_1(T) := \{\xi \in \mathbb{C}^N; \Re\langle \xi, \psi_1(T)\rangle = 0\}$, the solution Y of the linearised system around ψ_1,*

$$\begin{cases} i\frac{dY}{dt}(t) = H_0Y(t) - v(t)H_1\psi_1(t), \\ Y(0)=0, \end{cases} \tag{2.2.5}$$

with $v=\Gamma_L(Y_f)$ satisfies $Y(T)=Y_f$.

Proof of Proposition 2: For $v \in C^0([0,T],\mathbb{R})$, the solution of (2.2.5) is

$$Y(t) = i\int_0^t v(t)e^{-iH_0(t-s)}H_1\psi_1(s)ds, \forall t \in [0,T],$$

thus

$$Y(T) = \sum_{k=1}^N i\langle H_1\varphi_1,\varphi_k\rangle \int_0^T v(t)e^{i(\lambda_k-\lambda_1)t}dte^{-i\lambda_kT}\varphi_k.$$

The equality $Y(T)=Y_f$ is equivalent to $\langle Y(T),\varphi_k\rangle = \langle Y_f,\varphi_k\rangle, \forall k \in \{1,...,N\}$, i.e.

$$\int_0^T v(t)e^{i\omega_kt}dt = d_k := \frac{\langle Y_f,\varphi_{k+1}\rangle e^{i\lambda_{k+1}T}}{i\langle H_1\varphi_1,\varphi_{k+1}\rangle}, \forall k \in \{0,...,N-1\},$$

where $\omega_k := \lambda_{k+1}-\lambda_1$ for $k=0,...,N-1$. Let us prove that there exists a unique $(a_{-N+1},...,a_{N-1})^t \in \mathbb{C}^{2N-1}$ such that the function

$$v(t) := \sum_{k=-N+1}^{N-1} a_ke^{i\omega_kt}$$

solves

$$\int_0^T v(t)e^{i\omega_kt}dt = d_k, \forall k \in \{-N+1,...,N-1\}, \tag{2.2.6}$$

where $d_{-k} := \overline{d_k}$ for $k=1,...,N-1$. Indeed, (2.2.6) is equivalent to $Ga=d$ where $G \in \mathcal{M}_{2N-1}(\mathbb{C})$ is the Gram matrix of the family $(e^{-i\omega_kt})_{0\leqslant|k|\leqslant N-1}$ for the $L^2(0,T)$-scalar product and $a := (a_{-N+1}, ...a_{N-1})^t, d := (d_{-N+1}, ..., d_{N-1}) \in \mathbb{C}^{2N-1}$. Since $\omega_{-N+1} < ... < \omega_{N-1}$, G is invertible (see Corollary 1 in appendix), thus the solution is unique and $a = G^{-1}d$. Notice that $\tilde{a} := (\overline{a_{N-1}}, ..., \overline{a_{-N+1}})$ also solves $G\tilde{a} = d$ thus $a_{-k} = \overline{a_k}$ for $k = 0, ..., N$ and v is real valued. We have $\|a\| \leqslant \|G^{-1}\|\|d\| \leqslant C\|Y_f\|$ thus

$$\|v\|_{L^\infty(0,T)} \leqslant C\|Y_f\|,$$

which proves the continuity of $\Gamma_L : Y_f \mapsto v$. □

Proof of Theorem 3: We introduce the nonlinear map

$$\begin{array}{rccl} F: & C^0([0,T],\mathbb{R}) & \to & \mathbb{R}\varphi_1 + \mathbb{C}\varphi_2 + ... + \mathbb{C}\varphi_N \\ & u & \mapsto & X(T) - \Re\langle X(T), \psi_1(T)\rangle\psi_1(T), \end{array}$$

where $X := \Theta(u)$. Thanks to Proposition 1, F is C^1 and $dF(0).v = Y(T) - \Re\langle Y(T), \psi_1(T)\rangle$ where Y is the solution of (2.2.5) We have

$$\begin{aligned} \frac{d}{dt}\Re\langle Y(t), \psi_1(t)\rangle &= \Re\Big(\langle -iH_0Y(t) + iv(t)H_1\psi_1(t), \psi_1(t)\rangle + \langle Y(t), -iH_0\psi_1(t)\rangle\Big) \\ &= 0. \end{aligned}$$

Thus $\langle Y(t), \psi_1(t)\rangle \equiv 0$ and $dF(0).v = Y(T)$. Thus, the existence of continuous right inverse of $dF(0)$ is given by Proposition 2. We conclude by applying Theorem 4. □

2.3 Local controllability: the pathological case

We the assumption 2.2.1 is not satisfied, the local controllability around the ground state may still be proved, but there exists a minimal time, necessary for this local controllability. The proof uses power series expansions.

Chapter 3

Control in infinite dimension

In this chapter, we study the local exact controllability, around the ground state ψ_1, of the infinite dimensional system (1.1.1). Before stating the local controllability result, let us study the well posedness of the Cauchy problem.

3.1 Well posedness of the Cauchy-problem

In infinite dimension, the well posedness of the Cauchy-problem is more difficult because Cauchy-Lipschitz theorem cannot be applied. This is why we introduce the concept of weak solutions, whose existence is guaranteed by the next statement.

First, let us introduce some notations. We call $(e^{-iAt})_{t\in\mathbb{R}}$, the group of isometries of $L^2(0,1)$ generated by $-iA$, which is defined by the explicit expression

$$e^{-iAt}\varphi = \sum_{k=1}^{\infty}\langle\varphi,\varphi_k\rangle e^{-i\lambda_k t}\varphi_k, \tag{3.1.1}$$

where $\langle .,.\rangle$ denotes the $L^2(0,1)$-scalar product

$$\langle f,g\rangle = \int_0^1 f(x)\overline{g(x)}dx.$$

The spaces $H^s_{(0)}((0,1),\mathbb{C})$ are defined by

$$H^s_{(0)}(0,1) := D(A^{s/2}),$$

i.e.

$$H^0_{(0)}(0,1) = L^2(0,1),$$
$$H^1_{(0)}(0,1) = H^1_0(0,1) = \{\varphi\in H^1(0,1);\varphi(0)=\varphi(1)=0\},$$
$$H^2_{(0)}(0,1) = H^2\cap H^1_0(0,1) = \{\varphi\in H^2(0,1);\varphi(0)=\varphi(1)=0\},$$
$$H^3_{(0)}(0,1) = \{\varphi\in H^3(0,1);\varphi=\varphi''=0 \text{ at } x=0,1\},$$

$$H^4_{(0)}(0,1) = \{\varphi \in H^4(0,1); \varphi = \varphi'' = 0 \text{ at } x = 0,1\},$$

$$H^5_{(0)}(0,1) = \{\varphi \in H^5(0,1); \varphi = \varphi'' = \varphi^{(4)} = 0 \text{ at } x = 0,1\}, ...$$

In this section, we will work on the more general Cauchy problem

$$\begin{cases} i\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} - u(t)\mu(x)\psi - f(t,x), \ x \in (0,1), t \in \mathbb{R}_+, \\ \psi(t,0) = \psi(t,1) = 0, \\ \psi(0,x) = \psi_0(x). \end{cases} \tag{3.1.2}$$

We have the following existence result for weak solutions.

Proposition 3 *Let $s \in \{0,1,2\}$, $T > 0$, $u \in L^1((0,T),\mathbb{R})$, $\psi_0 \in H^s_{(0)}(0,1)$, $f \in L^1((0,T), H^s_{(0)}(0,1))$ and $C_\mu > 0$ be such that $\|\mu f\|_{H^s} \leqslant C_\mu \|f\|_{H^s}, \forall f \in H^s_{(0)}(0,1)$. There exists a unique weak solution of (3.1.2) such that*

$$\psi(0) = \psi_0, \tag{3.1.3}$$

i.e. a function $\psi \in C^0([0,T], H^s_{(0)}(0,1))$ such that the following equality holds in $H^s_{(0)}(0,1)$ for every $t \in [0,T]$,

$$\psi(t) = e^{-iAt}\psi_0 + i\int_0^t e^{-iA(t-\tau)}[u(\tau)\mu\psi(\tau) + f(\tau)]d\tau. \tag{3.1.4}$$

This weak solution satisfies

$$\|\psi\|_{C^0([0,T],H^s_{(0)})} \leqslant \Big(\|\psi_0\|_{H^s_{(0)}} + \|f\|_{L^1((0,T),H^s_{(0)})}\Big)e^{C_\mu\|u\|_{L^1(0,T)}}. \tag{3.1.5}$$

If $f \equiv 0$, then

$$\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}, \forall t \in [0,T]. \tag{3.1.6}$$

Proof of Proposition 3 : The existence and uniqueness result comes from a fixed point argument on the map F defined on $C^0([0,T], H^s_{(0)})$ by $F(\psi) := \xi$ where

$$\xi(t) := e^{-iAt}\psi_0 + i\int_0^t e^{-iA(t-\tau)}[u(\tau)\mu\psi(\tau) + f(\tau)]d\tau, \forall t \in [0,T].$$

F maps $C^0([0,T], H^s_{(0)})$ into itself because $\varphi \mapsto \mu\varphi$ and e^{-iAt} preserve $H^s_{(0)}(0,1)$. When $\|u\|_{L^1((0,T),\mathbb{R})}$ is small enough, then F is a contraction of $C^0([0,T], H^s_{(0)})$, because

$$\begin{aligned} \|\Theta(\psi_1)(t) - \Theta(\psi_2)(t)\|_{H^s_{(0)}} &= \Big\| \int_0^t e^{-iA(t-\tau)}u(\tau)\mu(\psi_1 - \psi_2)(\tau)d\tau \Big\|_{H^s_{(0)}} \\ &\leqslant \int_0^t \Big\|e^{-iA(t-\tau)}u(\tau)\mu(\psi_1 - \psi_2)(\tau)\Big\|_{H^s_{(0)}} d\tau \\ &= \int_0^t |u(\tau)| \Big\|\mu(\psi_1 - \psi_2)(\tau)\Big\|_{H^s_{(0)}} d\tau \\ &\leqslant C_\mu \|u\|_{L^1(0,T)} \|\psi_1 - \psi_2\|_{C^0([0,T],H^s_{(0)})}. \end{aligned}$$

Thus, F has a unique fixed point $\psi \in C^0([0,T], H^s_{(0)})$ that satisfies (3.1.4). If $\|u\|_{L^1((0,T),\mathbb{R})}$ is not small, one may use $0 = T_0 < T_1 < ... < T_n = T$ where, for $i = 0, ..., n-1$, $\|u\|_{L^1(T_i,T_{i+1})}$ is small enough so that the previous result holds on $[T_i, T_{i+1}]$, for $i = 0, ..., n-1$. Then we glue the solutions defined on $[T_0, T_1]$, $[T_1, T_2]$,...,$[T_{n-1}, T_n]$. We deduce from the equality (3.1.4) that

$$\|\psi(t)\|_{H^s_{(0)}} \leqslant \|\psi_0\|_{H^s_{(0)}} + \|f\|_{L^1((0,T),H^s_{(0)})} + \int_0^t |u(\tau)|C_\mu \|\psi(\tau)\|_{H^s_{(0)}} d\tau,$$

and Gronwall's Lemma gives (3.1.5). The proof of (3.1.6) will be done later. □

Remark 1 *This proof does not work with $s = 3$ because $\varphi \mapsto \mu\varphi$ does not preserve $H^3_{(0)}(0,1)$. Indeed, for $\varphi \in H^3_{(0)}(0,1)$ (i.e. $\varphi \in H^3(0,1)$ and $\varphi = \varphi'' = 0$ at $x = 0,1$), we have $(\mu\varphi)'' = 2\mu'\varphi'$ at $x = 0,1$ that may not vanish. One may prove that one can propagate the $H^3_{(0)}(0,1)$ regularity if one assumes more regularity on u, namely $u \in H^1((0,T),\mathbb{R})$ (see [10]).*

Now, let us prove the existence of $H^2_{(0)}$-strong solutions.

Proposition 4 *Let $T > 0$, $u \in C^0([0,T],\mathbb{R})$, $\psi_0 \in H^2_{(0)}(0,1)$ and $f \in L^1((0,T), H^2_{(0)}(0,1)) \cap C^0([0,T], L^2(0,1))$. The weak solution ψ of (3.1.2) is a strong solution i.e. $\psi \in C^0([0,T], H^2_{(0)}) \cap C^1([0,T], L^2)$, and the first equality of (3.1.2) holds in $L^2(0,1)$, for every $t \in [0,T]$.*

Proof of Proposition 4 : Under the assumptions of Proposition 4, the equality (3.1.4) ensures that $\psi \in C^1([0,T], L^2)$. Derivating (3.1.4) with respect to t, we get the first equality of (3.1.2) in L^2 for every $t \in [0,T]$. □

End of the proof of Proposition 3: Now, let us prove (3.1.6). When $u \in C^0([0,T],\mathbb{R})$, $\psi_0 \in H^2_{(0)}(0,1)$, and $f = 0$ then $\psi \in C^1([0,T], L^2)$ thus $t \mapsto \|\psi(t)\|^2_{L^2}$ is C^1 and

$$\frac{1}{2}\frac{d}{dt}\|\psi(t)\|^2_{L^2} = \Re\langle\psi(t), \frac{\partial\psi}{\partial t}\rangle = \Re\langle\psi(t), -\psi''(t) - u(t)\mu\psi(t)\rangle.$$

Since $\psi(t,0) \equiv \psi(t,1) \equiv 0$, integrations by parts prove that

$$\frac{d}{dt}\|\psi(t)\|^2_{L^2} = 0.$$

We have proved (3.1.6) when $u \in C^0([0,T],\mathbb{R})$ and $\psi_0 \in H^2_{(0)}$. A density argument using the continuity in $C^0([0,T], L^2)$ of the solutions with respect to $(u, \psi_0) \in L^2((0,T),\mathbb{R}) \times L^2(0,1)$ proves that (3.1.6) holds when $u \in L^2$ and $\psi_0 \in L^2$. □

3.2 Local controllability (non pathological case)

The following theorem may be proved (see [10]).

Theorem 5 *Let $T > 0$ and $\mu \in W^{3,\infty}((0,1),\mathbb{R})$ be such that*

$$\exists c_1, c_2 > 0 \text{ such that } \frac{c_1}{k^3} \leqslant |\langle \mu\varphi_1, \varphi_k\rangle| \leqslant \frac{c_2}{k^3}, \forall k \in \mathbb{N}^*. \tag{3.2.1}$$

There exists $\delta > 0$ and a Lipschitz map

$$\begin{array}{rccl} \Gamma: & \mathcal{V}_T & \rightarrow & H^1_0((0,T),\mathbb{R}) \\ & \psi_f & \mapsto & \Gamma(\psi_f) \end{array}$$

where

$$\mathcal{V}_T := \{\psi_f \in \mathcal{S} \cap H^5_{(0)}((0,1),\mathbb{C}); \|\psi_f - \psi_1(T)\|_{H^5} < \delta\},$$

such that, $\Gamma(\psi_1(T)) = 0$ and for every $\psi_f \in \mathcal{V}_T$, the solution of (1.1.1) with initial condition

$$\psi(0) = \varphi_1 \tag{3.2.2}$$

and control $u = \Gamma(\psi_f)$ satisfies $\psi(T) = \psi_f$.

The strategy for the proof of Theorem 5 in [10] is the same as in the previous chapter. However, the adaptation of Proposition 1 to the infinite dimensional case is rather difficult and cannot be detailed in this course. Thus, we only focus on the adaptation of Proposition 2 to the infinite dimensional case.

Proposition 5 *Let $T > 0$ and $\mu \in W^{3,\infty}((0,1),\mathbb{R})$ be such that (3.2.1) holds. There exists a continuous map $\Gamma_L : T_S\psi_1(T) \cap H^5_{(0)}(0,1) \rightarrow H^1_0((0,T),\mathbb{R})$, where $T_S\psi_1(T) := \{\varphi \in L^2(0,1); \Re\langle\varphi, \psi_1(T)\rangle = 0\}$, such that for every $\Psi_f \in T_S\psi_1(T) \cap H^5_{(0)}(0,1)$, the solution of the linearised system around ψ_1,*

$$\begin{cases} i\frac{\partial\Psi}{\partial t}(t,x) = -\frac{\partial^2\Psi}{\partial x^2}(t,x) - v(t)\mu(x)\psi_1(t,x), x \in (0,1), t \in [0,T], \\ \Psi(t,0) = \Psi(t,1) = 0, \\ \Psi(0,x) = 0, \end{cases} \tag{3.2.3}$$

with $v = \Gamma_L(\Psi_f)$, satisfies $\Psi(T) = \Psi_f$.

Proof of Proposition 5: For $v \in H^1_0((0,T),\mathbb{R})$, the solution of (3.2.3) is

$$\Psi(t,x) := i\int_0^t e^{-iA(t-s)} v(s)\mu\psi_1(s)ds, \forall t \in [0,T],$$

thus

$$\Psi(T) = \sum_{k=1}^{\infty} \langle \mu\varphi_1, \varphi_k\rangle \int_0^T v(t)e^{-i(\lambda_k - \lambda_1)t}dt e^{-i\lambda_k T}\varphi_k.$$

The equality $\Psi(T) = \psi_f$ is equivalent to $\langle \Psi(T), \varphi_k \rangle = \langle \Psi_f, \varphi_k \rangle, \forall k \in \mathbb{N}^*$, i.e.

$$\int_0^T v(t) e^{-i(\lambda_k - \lambda_1)t} dt = \frac{\langle \Psi_f, \varphi_k \rangle e^{-i\lambda_k T}}{\langle \mu\varphi_1, \varphi_k \rangle}, \forall k \in \mathbb{N}^*.$$

Performing an integration by parts, in which the boundary terms vanish because $v(0) = v(T) = 0$, we get the equivalence between $\psi(T) = \psi_f$ and the trigonometric moment problem

$$\begin{array}{l} \int_0^T \dot{v}(t) dt = d_0 := 0, \\ \int_0^T (T-t) v(t) dt = d_1 := \frac{\langle \Psi_f, \varphi_1 \rangle e^{-i\lambda_1 T}}{\langle \mu\varphi_1, \varphi_1 \rangle}, \\ \int_0^T \dot{v}(t) e^{-i(\lambda_k - \lambda_1)t} dt = d_k := \frac{i(\lambda_k - \lambda_1) \langle \Psi_f, \varphi_k \rangle e^{-i\lambda_k T}}{\langle \mu\varphi_1, \varphi_k \rangle}, \forall k \geqslant 2. \end{array}$$

Thanks to (3.2.1), the sequence $(d_k)_{k \in \mathbb{N}}$ belongs to $l^2(\mathbb{N}, \mathbb{C})$ when $\Psi_f \in H^5_{(0)}(0,1)$. The existence of a unique solution

$$\dot{v} \in \mathrm{Cl}_{L^2(0,T)} \Big(\mathrm{Span}(t, e^{\pm i(\lambda_k - \lambda_1)t}; k \in \mathbb{N}^*) \Big),$$

is a classical result from moment theory, proved in the next chapter. Moreover, this solution satisfies

$$\|\dot{v}\|_{L^2(0,T)} \leqslant C \|d\|_{l^2},$$

which gives the continuity of the map $\Gamma_L : \psi_f \mapsto v$. $\square$

Chapter 4

Trigonometric moment problems

The goal of this chapter is the proof of the following result.

Theorem 6 *Let $T > 0$ and $(\omega_k)_{k\in\mathbb{N}}$ be an increasing sequence of $[0, +\infty)$ such that $\omega_0 = 0$,*

$$\omega_{k+1} - \omega_k \to +\infty \text{ when } k \to +\infty.$$

There exists $C > 0$ such that, for every $d_{-1} \in \mathbb{R}$, $(d_k)_{k\in\mathbb{N}} \in l^2(\mathbb{N}, \mathbb{C})$ with $d_0 \in \mathbb{R}$, there exists a unique solution $u \in L^2((0, T), \mathbb{R})$ of the moment problem

$$\begin{aligned} &\int_0^T tu(t)dt = \tilde{d}, \\ &\int_0^T u(t)e^{i\omega_k t}dt = d_k, \forall k \in \mathbb{N}, \end{aligned} \tag{4.0.1}$$

that belongs to the closure of $Span\{t, e^{\pm i\omega_k t}; k \in \mathbb{N}\}$ in $L^2(0, T)$ and this solution satisfies

$$\|u\|_{L^2} \leqslant C \left(\sum_{k=-1}^{\infty} |d_k|^2 \right)^{1/2}.$$

Remark 2 *Note that the trigonometric moment problem (4.0.1) may be solved in any positive time $T > 0$, because the gap between two successive frequencies diverges.*

This chapter is organised as follows. In the two first sections, we recall classical results concerning abstract moment problems in Hilbert spaces (they come from [45]). The proofs are given for sake of completeness. In the third section, we focus on families of complex exponentials.

4.1 Family of vectors in Hilbert spaces

Let H be a separable Hilbert vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\Theta := (\xi_j)_{j\in\mathbb{N}}$ be a family of vectors of H with $\xi_j \neq 0, \forall j \in \mathbb{N}$.

Definition 3 *The family Θ is minimal in H if, for every $j \in \mathbb{N}$, $\xi_j \notin \overline{Span\{\xi_i; i \in \mathbb{N} - \{j\}\}}$.*

Proposition 6 *The family Θ is minimal in H if and only if there exists a biorthogonal family $\Theta' = (\xi'_j)_{j\in\mathbb{N}}$, i.e. Θ' is a family of vectors of H such that*

$$\langle \xi_i, \xi'_j \rangle = \delta_{i,j}, \forall i, j \in \mathbb{N}. \tag{4.1.1}$$

Proof of Proposition 6 : We assume Θ is minimal. For $j \in \mathbb{N}$, let v_j be the orthogonal projection of ξ_j over the closed vector space $\overline{\text{Span}\{\xi_i, i \neq j\}}$ i.e.

$$v_j \in \overline{\text{Span}\{\xi_i, i \neq j\}} \text{ and } \langle \xi_j - v_j, \xi_i \rangle = 0, \forall i \neq j.$$

Let

$$\xi'_j := \frac{\xi_j - v_j}{\|\xi_j - v_j\|^2}, \forall j \in \mathbb{N}^*.$$

Then, the families (ξ_j) and (ξ'_j) are biorthogonal.

Now, we assume that there exists a biorthogonal family $\Theta' = (\xi'_j)_{j\in\mathbb{N}}$. Let us assume that there exists $j \in \mathbb{N}$ such that $\xi_j \in \overline{\text{Span}\{\xi_i; i \in \mathbb{N} - \{j\}\}}$. Then (4.1.1) implies $\langle \xi_j, \xi'_j \rangle = 1$ and $\langle \xi_j, \xi'_j \rangle = 0$, which is a contradiction. □

Remark 3 *If Θ is minimal, then there exists a unique biorthogonal family Θ' such that $\Theta' \subset \overline{Span\{\xi_i; i \in \mathbb{N}\}}$. In the end of this appendix, when we speak about "the "biorthogonal family of Θ, we refer to this unique biorthogonal family in the closure of $Span\{\xi_i; i \in \mathbb{N}\}$.*

Definition 4 *The family Θ is a Riesz basis of $\overline{Span\Theta}$ if Θ is the image of some orthonormal family by an isomorphism.*

Remark 4 *It is clear that, if Θ is a Riesz basis of $\overline{Span\Theta}$, then Θ is minimal in H.*

Proposition 7 *(1) If Θ is a Riesz basis of $\overline{Span\Theta}$, then its biorthogonal family Θ' is also a Riesz basis of $\overline{Span\Theta}$.*

(2) Θ is a Riesz basis of $\overline{Span\Theta}$ if and only if there exists $C_1, C_2 \in (0, +\infty)$ such that, for every scalar sequence $(c_j)_{j\in\mathbb{N}}$ with finite support,

$$C_1 \left(\sum_{j=1}^{\infty} |c_j|^2 \right)^{1/2} \leqslant \left\| \sum_{j=1}^{\infty} c_j \xi_j \right\| \leqslant C_1 \left(\sum_{j=1}^{\infty} |c_j|^2 \right)^{1/2}. \tag{4.1.2}$$

Proof of Proposition 7 : (1) We assume Θ is a Riesz basis of $\overline{\mathrm{Span}\Theta}$. Let $\mathcal{H}$ be an Hilbert space, $(\zeta_j)_{j\in\mathbb{N}}$ be an orthonormal family of $\mathcal{H}$, $V : \mathcal{H} \to \overline{\mathrm{Span}\Theta}$ an isomorphism such that $\xi_j = V(\zeta_j), \forall j \in \mathbb{N}$. Then the adjoint operator $V^* : \overline{\mathrm{Span}\Theta} \to \mathcal{H}$ is also an isomorphism and we have $\xi'_j = (V^*)^{-1}(\zeta_j), \forall j \in \mathbb{N}$. Indeed, for every $j, k \in \mathbb{N}$,

$$\delta_{j,k} = \langle \xi_j, \xi'_k \rangle_H = \langle V(\zeta_j), \xi'_k \rangle_H = \langle \zeta_j, V^*(\xi'_k) \rangle_{\mathcal{H}}.$$

Thus Θ' is a Riesz basis of $\overline{\mathrm{Span}\Theta}$.

(2) We assume Θ is a Riesz basis of $\overline{\mathrm{Span}\Theta}$. Let $\mathcal{H}$ be an Hilbert space, $(\zeta_j)_{j\in\mathbb{N}}$ be an orthonormal family of $\mathcal{H}$, $V : \mathcal{H} \to \overline{\mathrm{Span}\Theta}$ an isomorphism such that $\xi_j = V(\zeta_j), \forall j \in \mathbb{N}$ and $(c_j)_{j\in\mathbb{N}}$ a scalar sequence with finite support. We have

$$\Big\| \sum_{j=0}^{\infty} c_j \xi_j \Big\| = \Big\| V \Big[\sum_{j=0}^{\infty} c_j \zeta_j \Big] \Big\| \leqslant \|V\| \Big\| \sum_{j=0}^{\infty} c_j \zeta_j \Big\| = \|V\| \left(\sum_{j=0}^{\infty} |c_j|^2 \right)^{1/2}$$

and

$$\left(\sum_{j=0}^{\infty} |c_j|^2 \right)^{1/2} = \Big\| \sum_{j=0}^{\infty} c_j \zeta_j \Big\| = \Big\| V^{-1} \Big[\sum_{j=0}^{\infty} c_j \xi_j \Big] \Big\| \leqslant \|V^{-1}\| \Big\| \sum_{j=0}^{\infty} c_j \xi_j \Big\|,$$

thus, we have (4.1.2) with $C_1 = 1/\|V^{-1}\|$ and $C_2 = \|V\|$.

Now, we assume that (4.1.2) holds. Then the linear map $V : l^2(\mathbb{N}, \mathbb{K}) \to \overline{\mathrm{Span}\Theta}$ defined by $V[(c_j)_{j\in\mathbb{N}}] = \sum_{j=0}^{\infty} c_j \xi_j$ is well defined and injective. Let $h \in \overline{\mathrm{Span}\Theta}$. There exists $(h_N)_{N\in\mathbb{N}}$ such that $h_N \to h$ in H when $N \to +\infty$ and for every $N \in \mathbb{N}$, there exists a sequence $c^{(N)} = (c_j^{(N)})_{j\in\mathbb{N}}$ with finite support such that $h_N = \sum_{j=0}^{\infty} c_j^{(N)} \xi_j$. Then $(h_N)_{N\in\mathbb{N}}$ is a Cauchy sequence in H, thus, thanks to (4.1.2), $(c^{(N)})_{N\in\mathbb{N}}$ is a Cauchy sequence in $l^2(\mathbb{N})$ and there exists $c = (c_j)_{j\in\mathbb{N}} \in l^2(\mathbb{N})$ such that $c^N \to c$ in $l^2(\mathbb{N})$. Then, (4.1.2) proves that $\sum_{j=0}^{\infty}(c_j - c_j^{(N)})\xi_j \to 0$ in H, i.e. $h = \sum_{j=0}^{\infty} c_j \xi_j$. We have proved that V is an isomorphism, thus Θ is a Riesz basis of $\overline{\mathrm{Span}\Theta}$. □

Remark 5 *We have proved that, if Θ is a Riesz basis of $\overline{\mathrm{Span}\Theta}$, then, for every $h \in \overline{\mathrm{Span}\Theta}$ there exists $c = (c_j)_{j\in\mathbb{N}} \in l^2(\mathbb{N}, \mathbb{K})$ such that $h = \sum_{j=0}^{\infty} c_j \xi_j$. Moreover, if Θ' and Θ are biorthogonal families, then necessarily $c_j = \langle h, \xi'_j \rangle, \forall j \in \mathbb{N}$. Thus, every $h \in \overline{\mathrm{Span}\Theta}$ can be decomposed in the following way*

$$h = \sum_{j=0}^{\infty} \langle h, \xi'_j \rangle \xi_j = \sum_{j=0}^{\infty} \langle h, \xi_j \rangle \xi'_j \tag{4.1.3}$$

*where the series converge in H and the coefficients $(\langle h, \xi'_j \rangle)_{j\in\mathbb{N}}$, $(\langle h, \xi_j \rangle)_{j\in\mathbb{N}}$, belong to $l^2(\mathbb{N}, \mathbb{K})$. For the last equality, we have used Proposition 7 **(1)**.*

4.2 Abstract moment problems

Now, we move to the investigation of abstract moment problems: given a scalar sequence $(d_j)_{j\in\mathbb{N}}$ is it possible to find $f \in H$ such that

$$\langle f, \xi_j \rangle = d_j, \forall j \in \mathbb{N}.$$

Let us introduce the operator

$$\begin{array}{rccc} J_\Theta : & H & \to & l^2(\mathbb{N}, \mathbb{K}) \\ & f & \mapsto & (\langle f, \xi_j \rangle)_{j\in\mathbb{N}} \end{array}$$

with domain

$$D_\Theta := \{f \in H; J_\Theta(f) \in l^2(\mathbb{N})\}.$$

It is clear that, if the family Θ is not complete in H, then the operator J_Θ has a non trivial null space $\overline{\text{Span}\Theta}^\perp$. This motivates the introduction of the operator $J_\Theta^0 := J_\Theta\Big|_{\overline{\text{Span}\Theta}}$.

Proposition 8 *The operator $J_\Theta^0 : \overline{Span\Theta} \to l^2(\mathbb{N}, \mathbb{K})$ is an isomorphism if and only if Θ is a Riesz basis of $\overline{Span\Theta}$.*

Proof of Proposition 8 : We assume $J_\Theta^0 : \overline{\text{Span}\Theta} \to l^2(\mathbb{N}, \mathbb{K})$ is an isomorphism. Let $(\zeta_j)_{j\in\mathbb{N}}$ be the canonical orthonormal basis of $l^2(\mathbb{N})$. Then, the family

$$\Big((J_\Theta^0)^{-1}(\zeta_j)\Big)_{j\in\mathbb{N}}$$

is a Riesz basis of $\overline{\text{Span}\Theta}$. Moreover, it is the biorthogonal family to Θ in $\overline{\text{Span}\Theta}$. Thanks to Proposition 7 **(1)**, Θ is also a Riesz basis of $\overline{\text{Span}\Theta}$.

We assume Θ is a Riesz basis of $\overline{\text{Span}\Theta}$. Thanks to the Remark 5, it is clear that $J_\Theta^0 : \overline{\text{Span}\Theta} \to l^2(\mathbb{N}, \mathbb{K})$ is an isomorphism. □

4.3 Ingham inequality for complex exponentials

The goal of this section is the proof of the Ingham inequality (4.1.2) for families of complex exponentials $(\xi_j = e^{i\omega_j t})_{j\in\mathbb{Z}}$. First, let us prove the easiest inequality.

Proposition 9 *Let $T > 0$ and $(\omega_k)_{k\in\mathbb{Z}}$ be an increasing sequence such that*

$$\omega_k - \omega_{k-1} \geqslant \gamma > 0, \forall k \in \mathbb{Z}. \tag{4.3.1}$$

There exists $C = C(T, \gamma) > 0$ such that, for every $(a_k)_{k\in\mathbb{Z}} \in \mathbb{C}^\mathbb{Z}$ with finite support,

$$\int_{-T}^{T} \Big| \sum_k a_k e^{i\omega_k t} \Big|^2 dt \leqslant C \sum_k |a_k|^2. \tag{4.3.2}$$

Remark 6 *Notice that, for this easiest inequality, no relation is needed between T and γ.*

Proof of Proposition 9:

First case: We assume $T < \pi$. We introduce the functions

$$k(t) := \begin{cases} \cos(t/2), \text{ if } |t| \leqslant \pi, \\ 0, \text{ if } |t| > \pi, \end{cases}$$

$$K(\omega) := \int_{\mathbb{R}} k(t)e^{-i\omega t}dt = \begin{cases} \frac{4\cos(\pi\omega)}{1-4\omega^2}, \text{ if } \omega \neq \pm 1/2, \\ \pi, \text{ if } \omega = \pm 1/2, \end{cases} \tag{4.3.3}$$

$$f(t) := \sum_{j=-N}^{N} a_j e^{-i\omega_j t}.$$

Since $T \in (0, \pi)$ then $k(t) \geqslant k(T) > 0, \forall t \in [-T, T]$, thus, we have

$$\int_{-T}^{T} |f(t)|^2 dt \leqslant \frac{1}{k(T)} \int_{-T}^{T} k(t)|f(t)|^2 dt \leqslant \frac{1}{k(T)} \int_{-\pi}^{\pi} k(t)|f(t)|^2 dt.$$

We have

$$\begin{aligned} \int_{-\pi}^{\pi} k(t)|f(t)|^2 dt &= \sum_k \sum_j a_k \overline{a_j} K(\omega_k - \omega_j) \\ &= \sum_k |a_k|^2 K(0) + \sum_k \sum_{j \neq k} a_k \overline{a_j} K(\omega_k - \omega_j) \\ &\leqslant 4 \sum_k |a_k|^2 + \sum_k \sum_{j \neq k} \frac{|a_k|^2 + |a_j|^2}{2} |K(\omega_k - \omega_j)| \\ &= \sum_k |a_k|^2 \Big(4 + \frac{1}{2} \sum_{j \neq k} |K(\omega_k - \omega_j)|\Big) + \frac{1}{2} \sum_k \sum_{j \neq k} |a_j|^2 |K(\omega_k - \omega_j)| \\ &= \sum_k |a_k|^2 \Big(4 + \sum_{j \neq k} |K(\omega_k - \omega_j)|\Big). \end{aligned}$$

Thanks to (4.3.3), there exists $C > 0$ such that $|K(\omega)| \leqslant C/(1+\omega^2), \forall \omega \in \mathbb{R}$. Using (4.3.1), we get, for every $k \in \mathbb{Z}$

$$\sum_{j \neq k} |K(\omega_k - \omega_j)| \leqslant \sum_{j \neq k} \frac{C}{1 + \gamma^2 (k-j)^2} \leqslant 2C \sum_{n=1}^{\infty} \frac{1}{1 + \gamma^2 n^2}.$$

Therefore, we have (4.3.2) with

$$C(T, \gamma) := \frac{1}{k(T)} \Big(4 + 2C \sum_{n=1}^{\infty} \frac{1}{1 + \gamma^2 n^2}\Big).$$

Second case: We assume $T \geqslant \pi$. Thanks to the change of variable $t = 2Tx/\pi$, we get

$$\int_{-T}^{T} |f(t)|^2 dt = \frac{2T}{\pi} \int_{-\pi/2}^{\pi/2} \Big| \sum_k a_k e^{i \frac{2T\omega_k}{\pi} x} \Big|^2 dx \leqslant \frac{2T}{\pi} C\left(\frac{\pi}{2}, \frac{2T\gamma}{\pi}\right) \sum_k |a_k|^2. \Box$$

Now, let us prove the more difficult inequality. The following result comes from [34, Theorem 1.2.9, page 59].

Theorem 7 *Let $(\omega_k)_{k\in\mathbb{Z}}$ an increasing sequence such that (4.3.1) holds and $T>0$ be such that $T\gamma>\pi$. For every $(a_k)_{k\in\mathbb{Z}}\in\mathbb{C}^{\mathbb{Z}}$ with finite support, we have*

$$\sum_k |a_k|^2 \leqslant \frac{A(\epsilon)}{2T}\int_{-T}^{T}\Big|\sum_k a_k e^{-i\omega_k t}\Big|^2 dt, \tag{4.3.4}$$

where $\epsilon := \gamma T-\pi$ and

$$A(\epsilon) := \frac{\pi(\pi+\epsilon)^2}{2\epsilon(2\pi+\epsilon)}.$$

Proof of Theorem 7: First, let us emphasise that it is sufficient to prove Theorem 7 with $T=\pi$. Indeed, let us assume that Theorem 7 is known for $T=\pi$ and let us prove it for arbitrary $T>0$. With the change of variable $t=Tx/\pi$, we get

$$\frac{1}{2T}\int_{-T}^{T}|f(t)|^2dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\sum_k a_k e^{-i\frac{\omega_k T}{\pi}x}\Big|^2 dx$$

and

$$\frac{\omega_{k+1}T}{\pi}-\frac{\omega_k T}{\pi} > \gamma\frac{T}{\pi} = \frac{\pi+\epsilon}{\pi} > 1,$$

thus, applying Theorem 7 with $T=\pi$, we get

$$\frac{1}{2T}\int_{-T}^{T}|f(t)|^2dt \geqslant \frac{1}{A(\epsilon)}\sum_k |a_k|^2.$$

Now, let us prove Theorem 7 with $T=\pi$ Let k, K and f be as in the previous proof. Working as in the previous proof, we get

$$\int_{-\pi}^{\pi}|f(t)|^2dt \geqslant \int_{-\pi}^{\pi}k(t)|f(t)|^2dt \geqslant \sum_k |a_k|^2\Big(4-\sum_{j\neq k}|K(\omega_k-\omega_j)|\Big).$$

For every $j,k\in\mathbb{Z}$ with $j\neq k$, we have

$$|\omega_j-\omega_k| \geqslant |j-k|\gamma > 1.$$

Thus

$$|K(\omega_j-\omega_k)| \leqslant \frac{4}{4(j-k)^2\gamma^2-1} < \frac{4}{\gamma^2}\frac{1}{4(j-k)^2-1}.$$

We deduce that, for every $k\in\mathbb{Z}$, we have

$$\begin{aligned}\sum_{j\neq k}|K(\omega_j-\omega_k)| &\leqslant \tfrac{2}{\gamma^2}\textstyle\sum_{j\neq k}\frac{2}{4(j-k)^2-1}\\ &\leqslant \tfrac{4}{\gamma^2}\textstyle\sum_{n=1}^{\infty}\frac{2}{4n^2-1}\\ &= \tfrac{4}{\gamma^2}\textstyle\sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\\ &= \tfrac{4}{\gamma^2}.\end{aligned}$$

As a result, we obtain (4.3.4) for $T = \pi$. □

We deduce from Proposition 9 and Theorem 7 the following result.

Theorem 8 *Let $T > 0$, $\gamma > 0$, and $(\omega_k)_{k\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ be an increasing sequence. We assume (4.3.1) and*

$$T > \frac{2\pi}{\gamma}. \tag{4.3.5}$$

Then, there exists c, C depending only on γ and T such that, for every $(a_k)_{k\in\mathbb{Z}}$ with finite support,

$$c\sum_{k\in\mathbb{Z}} |a_k|^2 \leqslant \int_0^T \Big|\sum_{k\in\mathbb{Z}} a_k e^{i\omega_k t}\Big|^2 dt \leqslant C\sum_{k\in\mathbb{Z}} |a_k|^2. \tag{4.3.6}$$

Proof of Theorem 8: One just needs to perform the change of variable $t = x + T/2$ is order to get integrals over $(-T/2, T/2)$ as in Proposition 9 and Theorem 7. □

The following adaptation is due Haraux [28, Theorem 4, page 461].

Theorem 9 *Let $T > 0$, $N \in \mathbb{N}^*$, $\gamma > 0$, $\rho > 0$ and $(\omega_k)_{k\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ be an increasing sequence. We assume*

$$\omega_{k+1} - \omega_k \geqslant \gamma, \forall |k| \geqslant N, \tag{4.3.7}$$

$$\omega_{k+1} - \omega_k \geqslant \rho, \forall k \in \mathbb{Z}, \tag{4.3.8}$$

$$T > \frac{2\pi}{\gamma}. \tag{4.3.9}$$

Then, there exists c, C depending only on γ, ρ, N and T such that, for every $(a_k)_{k\in\mathbb{Z}}$ with finite support,we have (4.3.6).

Proof of Theorem 9: The second inequality in (4.3.6) is a consequence of the equivalent one in Theorem 8 because we add only a finite number of frequencies. More precisely, using the triangular inequality and Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\left(\int_0^T \Big|\sum_{k\in\mathbb{Z}} a_k e^{i\omega_k t}\Big|^2 dt\right)^{1/2} \\
\leqslant\ & \left(\int_0^T \Big|\sum_{|k|<N} a_k e^{i\omega_k t}\Big|^2 dt\right)^{1/2} + \left(\int_0^T \Big|\sum_{|k|\geqslant N} a_k e^{i\omega_k t}\Big|^2 dt\right)^{1/2} \\
\leqslant\ & \sum_{|k|<N} |a_k|\sqrt{T} + \sqrt{C}\left(\sum_{|k|\geqslant N} |a_k|^2\right)^{1/2} \\
\leqslant\ & \sqrt{TN}\left(\sum_{|k|<N} |a_k|^2\right)^{1/2} + \sqrt{C}\left(\sum_{|k|\geqslant N} |a_k|^2\right)^{1/2} \\
\leqslant\ & (\sqrt{TN} + \sqrt{C})\left(\sum_{k\in\mathbb{Z}} |a_k|^2\right)^{1/2}.
\end{aligned}$$

We prove the first inequality in (4.3.6) by induction on the number $p \in \mathbb{N}^*$ of indices $n \in \mathbb{Z}$ such that $\omega_{n+1} - \omega_n < \gamma$. If $p = 0$, the result is given by Theorem 8. Let $p \in \mathbb{N}^*$. We may assume that $\omega_1 - \omega_0 < \gamma$. By assumption, the result is known for functions of the form

$$\sum_{n \neq 0} a_k e^{i\omega_k t}.$$

We can assume that $\omega_0 = 0$ (otherwise, replace ω_k by $\omega_k - \omega_0$). Let $\epsilon > 0$ be such that

$$T' := T - \epsilon > \frac{2\pi}{\gamma},$$

and

$$f(t) := \sum_{k \in \mathbb{Z}} a_k e^{i\omega_k t}.$$

We have

$$f(t+\eta) - f(t) = \sum_{k \neq 0} a_k (e^{i\omega_k \eta} - 1) e^{i\omega_k t}, \forall \eta \in [0, \epsilon].$$

Integrating this relation over $(0, \epsilon)$, we get

$$\int_0^\epsilon [f(t+\eta) - f(t)] d\eta = \sum_{k \neq 0} a_k \left(\frac{e^{i\omega_k \epsilon} - 1}{i\omega_k} - \epsilon \right) e^{i\omega_k t}.$$

Thanks to (4.3.8) and $\omega_0 = 0$, we have

$$\frac{\omega_k \epsilon}{2} \geqslant \frac{\rho \epsilon}{2},$$

thus, there exists $\delta = \delta(\rho) > 0$ such that

$$|e^{i\omega_k \epsilon} - 1| = 2 \Big| \sin\left(\frac{\omega_k \epsilon}{2} \right) \Big| \leqslant \epsilon |\omega_k| (1 - \delta), \forall k \in \mathbb{Z},$$

$$\Big| \frac{e^{i\omega_k \epsilon} - 1}{i\omega_k} - \epsilon \Big| \geqslant \epsilon - \Big| \frac{e^{i\omega_k \epsilon} - 1}{i\omega_k} \Big| \geqslant \epsilon - (1-\delta)\epsilon = \delta \epsilon.$$

Applying the induction assumption, we get

$$\int_0^{T'} \Big| \int_0^\epsilon [f(t+\eta) - f(t)] d\eta \Big|^2 dt \geqslant C \sum_{k \neq 0} |a_k|^2 \Big| \frac{e^{i\omega_k \epsilon} - 1}{i\omega_k} - \epsilon \Big|^2 \geqslant C \epsilon^2 \delta^2 \sum_{k \neq 0} |a_k|^2. \tag{4.3.10}$$

Moreover, thanks to Cauchy-Schwarz inequality, we have

$$\begin{aligned} \int_0^{T'} \Big| \int_0^\epsilon [f(t+\eta) - f(t)] d\eta \Big|^2 dt &\leqslant \int_0^{T'} \epsilon \int_0^\epsilon |f(t+\eta) - f(t)|^2 d\eta dt \\ &\leqslant 2\epsilon \int_0^\epsilon \int_0^{T'} |f(t+\eta)|^2 + |f(t)|^2 dt d\eta \\ &\leqslant 2\epsilon \int_0^\epsilon 2 \int_0^T |f(t)|^2 dt d\eta \\ &\leqslant 4\epsilon^2 \int_0^T |f(t)|^2 dt. \end{aligned} \tag{4.3.11}$$

Thus, combining (4.3.10) and (4.3.11), we get

$$\sum_{k\neq 0}|a_k|^2 \leqslant \frac{4}{C\delta}\int_0^T |f(t)|^2 dt. \tag{4.3.12}$$

In order to conclude, it is sufficient to prove the existence of $c > 0$ such that

$$|a_0| \leqslant c\|f\|_{L^2(0,T)}. \tag{4.3.13}$$

We have

$$a_0 - f(t) = \sum_{k\neq 0} a_k e^{i\omega_k t} := g(t).$$

We know that

$$\int_0^T |f(t) - a_0|^2 dt = \int_0^T |g(t)|^2 \leqslant C_1 \sum_{k\neq 0}|a_k|^2. \tag{4.3.14}$$

From (4.3.12), (4.3.14), we get

$$\begin{aligned}
\sqrt{T}|a_0| &= \|a_0\|_{L^2(0,T)} \\
&\leqslant \|a_0 - f\|_{L^2(0,T)} + \|f\|_{L^2(0,T)} \\
&\leqslant \sqrt{C_1}\left(\sum_{k\neq 0}|a_k|^2\right)^{1/2} + \|f\|_{L^2(0,T)} \\
&\leqslant \left(\sqrt{\frac{4C_1}{C\delta}} + 1\right)\|f\|_{L^2(0,T)},
\end{aligned}$$

which proves (4.3.13).□

Proof of Theorem 6: Let $\omega_k := -\omega_{-k}$, for every $k \in \mathbb{Z}$ with $k < 0$. From Theorem 9, $\Theta := (e^{i\omega_k t})_{k\in\mathbb{Z}}$ is a Riesz basis of $\mathrm{Cl}_{L^2(0,T)}\mathrm{Span}\Theta$.

First step: We prove that the family $\widetilde{\Theta} := \{t, e^{i\omega_k t}; k \in \mathbb{Z}\}$ *is minimal in* $L^2(0,T)$.

Working by contradiction, we assume that $\widetilde{\Theta}$ is not minimal in $L^2(0,T)$. Then, necessarily

$$t \in \mathrm{Cl}_{L^2(0,T)}\mathrm{Span}\Theta. \tag{4.3.15}$$

With successive integrations, we get

$$t^j \in \mathrm{Cl}_{C^0[0,T]}\left(\mathrm{Span}\widetilde{\Theta}\right), \forall j \in \mathbb{N} \text{ with } j \geqslant 2.$$

The Stone Weierstrass theorem ensures that $\{1, t^j; j \in \mathbb{N}, j \geqslant 2\}$ is dense in $C^0([0,T],\mathbb{C})$, thus, it is also dense in $L^2(0,T)$. From (4.3.15), we deduce that $\mathrm{Span}\Theta$ is dense in $L^2(0,T)$. This is a contradiction, because, thanks to

Theorem 9, for every $\omega \in \mathbb{R} - \{\omega_k, k \in \mathbb{Z}\}$, the family $\{e^{i\omega t}, e^{i\omega_k t}; k \in \mathbb{Z}\}$ is minimal, i.e.

$$e^{i\omega t} \notin \mathrm{Cl}_{L^2(0,T)}\mathrm{Span}\Theta.$$

Second step: We conclude.

For $k < 0$, we define $d_k := \overline{d_{-k}}$. Let $\{\tilde{\xi}, \xi_k; k \in \mathbb{Z}\}$ be the biorthogonal family to $\{t, e^{i\omega_k t}; k \in \mathbb{Z}\}$. From Theorem 9, there exists $C > 0$ (independent of $(d_k)_{k\in\mathbb{Z}}$) and a unique solution $v \in \mathrm{Cl}_{L^2(0,T)}\mathrm{Span}\Theta$ of

$$\int_0^T v(t)e^{i\omega_k t}dt = d_k, \forall k \in \mathbb{Z}$$

and it satisfies

$$\|v\|_{L^2(0,T)} \leqslant C\left(\sum_{k\in\mathbb{Z}}|d_k|^2\right)^{1/2}.$$

The uniqueness guarantees that v is real valued. Let us define

$$u := v + \Big(\tilde{d} - \int_0^T tv(t)dt\Big)\tilde{\xi}.$$

Then, u is real valued (because $\tilde{\xi}$ is), $u \in \mathrm{Cl}_{L^2(0,T)}\mathrm{Span}\widetilde{\Theta}$, u solves (4.0.1) and

$$\begin{aligned}
\|u\|_{L^2} &\leqslant \|v\|_{L^2} + \Big(|\tilde{d}| + \Big|\textstyle\int_0^T tv(t)dt\Big|\Big)\|\tilde{\xi}\|_{L^2} \\
&\leqslant \|v\|_{L^2}\Big(1 + \sqrt{\tfrac{T^3}{3}}\|\tilde{\xi}\|_{L^2}\Big) + |\tilde{d}|\|\tilde{\xi}\|_{L^2} \\
&\leqslant \left(C\Big(1 + \sqrt{\tfrac{T^3}{3}}\|\tilde{\xi}\|_{L^2}\Big) + \|\tilde{\xi}\|_{L^2}\right)\Big(|\tilde{d}|^2 + \textstyle\sum_{k\in\mathbb{Z}}|d_k|^2\Big)^{1/2}.
\end{aligned}$$

The uniqueness comes from the fact that u is chosen in $\mathrm{Cl}_{L^2(0,T)}\mathrm{Span}\widetilde{\Theta}$. $\square$

Chapter 5

Stabilisation in finite dimension

5.1 Heuristic

For a given initial condition $X_0 \in \mathbb{C}^N$ with $\|X_0\| = 1$, we search a control $u : [0, +\infty) \to \mathbb{R}$ such that the solution of (1.1.5) converges to the ground state when $t \to +\infty$, i.e.

$$|\langle X(t), \varphi_1 \rangle| \to 1, \text{ when } t \to +\infty. \tag{5.1.1}$$

Let us introduce the Lyapunov function

$$\mathcal{L}(X) := |\langle X, \varphi_1 \rangle|^2.$$

For a solution of (1.1.5), we have

$$\begin{aligned} \frac{d\mathcal{L}(X)}{dt} &= 2\Re\Big(\langle [-iH_0 + iu(t)H_1]X(t), \varphi_1 \rangle \overline{\langle X(t), \varphi_1 \rangle} \Big) \\ &= 2\Re\Big(-i\lambda_1 |\langle X(t), \varphi_1 \rangle|^2 + iu(t) \langle H_1 X(t), \varphi_1 \rangle \overline{\langle X(t), \varphi_1 \rangle} \Big) \\ &= -2u(t)\Im\Big(\langle H_1 X(t), \varphi_1 \rangle \overline{\langle X(t), \varphi_1 \rangle} \Big). \end{aligned}$$

Therefore, if we chose

$$u(X) = -\Im\Big(\langle H_1 X, \varphi_1 \rangle \overline{\langle X, \varphi_1 \rangle} \Big) \tag{5.1.2}$$

then, $t \mapsto \mathcal{L}(X(t))$ is not decreasing because

$$\frac{d\mathcal{L}(X)}{dt} = u(X)^2 \geqslant 0. \tag{5.1.3}$$

Thus, we can expect to have (5.1.1). This convergence is proved in section 5.3. (Note also that, in general, closed loop controls are more robust (less sensitive to perturbations) than open loop controls.) First, let us prove the existence of solutions to the closed loop system.

5.2 Well posedness of the closed loop system

Proposition 10 *Let $X_0 \in \mathbb{C}^N$ with $\|X_0\| = 1$. There exists a unique solution $X \in C^1([0,+\infty),\mathbb{C}^N)$ of the closed loop system*

$$\begin{cases} i\frac{dX}{dt} = H_0 X - u(X) H_1 X, \\ X(0) = X_0, \end{cases} \tag{5.2.1}$$

where $u(X)$ is defined by (5.1.2). Moreover, these solutions are continuous with respect to the initial conditions: for every $T > 0$, for every $X_0^1, X_0^2 \in \mathbb{C}^N$ with $\|X_0^1\| = \|X_0^2\| = 1$, we have

$$\|X^1 - X^2\|_{L^\infty(0,T)} \leqslant \|X_0^1 - X_0^2\| e^{4\|H_1\|^2 T}. \tag{5.2.2}$$

Proof of Proposition 10: The function $f(X) := -iH_0X + iu(X)H_1X$ is locally Lipschitz on $\mathbb{C}^N$, thus the Cauchy-Lipschitz theorem ensures the existence of a unique maximal solution of (5.2.1) defined on an interval $[0,T']$ for some $T' \in (0,T]$: $X \in C^1([0,T'),\mathbb{C}^N)$. Moreover, either $T' = T$, or $T' < T$ and $\limsup_{t\to T'} \|X(t)\| = +\infty$. However, we have $\|X(t)\| = 1, \forall t \in [0,T')$, thus $T' = T$. From

$$\begin{cases} i\frac{d(X^1-X^2)}{dt} = [H_0 - u(X^1)H_1](X^1 - X^2) - [u(X^1) - u(X^2)]H_1X^2, \\ (X^1 - X^2)(0) = X_0^1 - X_0^2, \end{cases}$$

we get,

$$\begin{aligned} & \tfrac{1}{2}\tfrac{d}{dt}\|X^1 - X^2\|^2 \\ = \;& \Re\langle -i[H_0 - u(X^1)H_1](X^1 - X^2) + i[u(X^1) - u(X^2)]H_1X^2, X^1 - X^2\rangle \\ = \;& -[u(X^1) - u(X^2)]\Im\langle H_1X^2, X^1 - X^2\rangle \\ \leqslant \;& |u(X^1) - u(X^2)|\|H_1\|\|X^1 - X^2\|. \end{aligned}$$

Moreover, we have

$$\begin{aligned} & |u(X^1) - u(X^2)| \\ = \;& |\Im\Big(\langle H_1(X^1 - X^2), \varphi_1\rangle\overline{\langle X^1, \varphi_1\rangle}\Big) + \Im\Big(\langle H_1X^2, \varphi_1\rangle\overline{\langle (X^1 - X^2), \varphi_1\rangle}\Big)| \\ \leqslant \;& 2\|H_1\|\|X^1 - X^2\|. \end{aligned}$$

Therefore, we have

$$\frac{d}{dt}\|X^1 - X^2\|^2 \leqslant 4\|H_1\|^2\|X_1 - X_2\|^2,$$

which gives (5.2.2). □

5.3 Convergence: the non pathological case

The goal of this section is the proof of the convergence of the solutions of the closed loop system to the ground state. This result is taken from [38].

Theorem 10 *We assume*

$$\langle H_1\varphi_1, \varphi_k\rangle \neq 0, \forall k \in \{2, ..., N\}. \tag{5.3.1}$$

Then, for every $X_0 \in \mathbb{C}^N$ *with* $\|X_0\| = 1$ *and* $\langle X_0, \varphi_1\rangle \neq 0$*, the solution of (5.2.1) satisfies*

$$\mathcal{L}(X(t)) \to 1, \text{ when } t \to +\infty.$$

Remark 7 *The assumption (5.3.1) means that the* $(N-1)$ *last coefficients of the first column of* H_1 *are all different from zero.*

The proof of Theorem 10 relies on LaSalle invariance principle. First, we focus on the invariant set.

Proposition 11 *We assume (5.3.1). Let* $X_0 \in \mathbb{C}^N$ *with* $\|X_0\| = 1$, $\langle X_0, \varphi_1\rangle \neq 0$*, and* X *be the solution of (5.2.1). We assume that* $t \mapsto \mathcal{L}(X(t))$ *is constant. Then, there exists* $\theta \in [0, 2\pi)$ *such that* $X(t) = e^{i\theta}\psi_1(t)$*.*

Proof of Proposition 11: From (5.1.3), we get $u(X) \equiv 0$. We have

$$i\frac{dX}{dt}(t) = H_0 X(t),$$

thus

$$X(t) = \sum_{k=1}^{N} \langle X_0, \varphi_k\rangle e^{-i\lambda_k t}\varphi_k.$$

Thanks to (5.1.2), $u(X) \equiv 0$ gives

$$\Im\Big(\sum_{k=1}^{N} \langle X_0, \varphi_k\rangle\langle H_1\varphi_k, \varphi_1\rangle e^{-i\lambda_k t}\overline{\langle X_0, \varphi_1\rangle} e^{i\lambda_1 t}\Big) = 0, \forall t \in [0, +\infty),$$

or, equivalently,

$$\sum_{|k|=2}^{N} d_k e^{-i\omega_k t} = 0, \forall t \in [0, +\infty).$$

where $d_k := \langle H_1\varphi_k, \varphi_1\rangle\langle X_0, \varphi_k\rangle\overline{\langle X_0, \varphi_1\rangle}$, $d_{-k} := -\overline{d_k}$, $\omega_k := \lambda_k - \lambda_1$, $\omega_{-k} := -\omega_k$, $\forall k \in \{2, ..., N\}$. Since $\omega_{-N} < ... < \omega_{-2} < \omega_2 < ... < \omega_N$, the family $(e^{-i\omega_k t})_{2 \leqslant |k| \leqslant N}$ is linearly independent (see Corollary 1 in appendix) Therefore, $d_k = 0$ for $k = 2, ..., N$. Thanks to (5.3.1), we have $\langle X_0, \varphi_k\rangle\overline{\langle X_0, \varphi_1\rangle} = 0$ for $k = 2, ..., N$. Since $\langle X_0, \varphi_1\rangle \neq 0$, then $\langle X_0, \varphi_k\rangle = 0$ for $k = 2, ..., N$, thus, $X_0 = e^{i\theta}\varphi_1$ and $X(t) = e^{i\theta}\psi_1(t)$, for some $\theta \in [0, 2\pi)$.

□

Proof of Theorem 10: Let $X_0 \in \mathbb{C}^N$ with $\|X_0\| = 1$, $\langle X_0, \varphi_1\rangle \neq 0$ and X be the solution of (5.2.1). The function $t \mapsto \mathcal{L}(X(t))$ is not decreasing, thus, there exists $L \in [\mathcal{L}(X_0), 1] \subset (0, 1]$ such that

$$\lim_{t\to+\infty} \mathcal{L}(X(t)) = L.$$

Let $(t_n)_{n\in\mathbb{N}}$ be an increasing sequence of $[0, +\infty)$ such that $t_n \to +\infty$. The sequence $(X(t_n))_{n\in\mathbb{N}}$ is bounded in $\mathbb{C}^N$ (indeed $\|X(t)\| = 1, \forall t \in [0, +\infty)$), thus, there exist $X_0^\infty \in \mathbb{C}^N$ with $\|X_0^\infty\| = 1$ and an extraction s such that $X(t_{s(n)}) \to X_0^\infty$. Let X^∞ be the solution of the closed loop system (5.2.1) with initial condition X_0^∞. Then $X(t_{s(n)} + t) \to X^\infty(t)$ when $n \to +\infty$ for every $t \in [0, +\infty)$ (see (5.2.2)). Thus,

$$\mathcal{L}[X^\infty(t)] = \lim_{n\to+\infty} \mathcal{L}[X(t_{s(n)})] = L, \forall t \in [0, +\infty).$$

The map $t \mapsto \mathcal{L}[X^\infty(t)]$ is constant. Moreover, $\langle X_0^\infty, \varphi_1\rangle \neq 0$, because $\mathcal{L}[X_0^\infty] = L \in (0, 1]$, and $\|X_0^\infty\| = 1$. Thus, Proposition 11 ensures that $X_0^\infty = e^{i\theta}\varphi_1$ for some $\theta \in [0, 2\pi)$. In conclusion, we have $L = \mathcal{L}[X_0^\infty] = 1$. □

5.4 Convergence: a pathological case

We refer to the reference [15] for an adaptation of the previous strategy, to situations where the assumption (5.3.1) is not satisfied.

Chapter 6

Stabilisation in infinite dimension

6.1 Heuristic

First, let us perform the same heuristic as in the previous chapter. We introduce the Lyapunov function

$$\mathcal{L}(\psi) := |\langle \psi, \varphi_1 \rangle|^2. \tag{6.1.1}$$

Let ψ be a solution of (1.1.1). If $u \in C^0([0,T],\mathbb{R})$, then $\psi \in C^1([0,T],L^2(0,1))$, thus $t \mapsto |\langle \psi(t), \varphi_1|^2$ is differentiable and we have

$$\begin{aligned} \tfrac{d\mathcal{L}(\psi)}{dt}(t) &= 2\Re\Big(\langle \tfrac{\partial \psi}{\partial t}(t), \varphi_1\rangle \overline{\langle \psi(t), \varphi_1\rangle}\Big) \\ &= 2\Re\Big(\langle -i\psi''(t) + iu(t)\mu\psi(t), \varphi_1\rangle \overline{\langle \psi(t), \varphi_1\rangle}\Big). \end{aligned}$$

Since $\psi \in H^2(0,1)$ and $\psi(t,x) = \varphi_1(x) = 0$ at $x = 0, 1$, integrations by parts give $\langle -i\psi'', \varphi_1\rangle = -i\lambda_1\langle \psi, \varphi_1\rangle$, thus, we get

$$\frac{d\mathcal{L}(\psi)}{dt}(t) = -u(t)\Im\Big(\langle \mu\psi(t), \varphi_1\rangle \overline{\langle \psi(t), \varphi_1\rangle}\Big).$$

Let us introduce

$$u(\psi) := -\Im\Big(\langle \mu\psi(t), \varphi_1\rangle \overline{\langle \psi(t), \varphi_1\rangle}\Big), \tag{6.1.2}$$

then

$$\frac{d\mathcal{L}(\psi)}{dt} = u(\psi)^2. \tag{6.1.3}$$

6.2 Well posedness of the closed loop system

Proposition 12 *Let $\psi_0 \in L^2(0,1)$ be such that $\|\psi_0\|_{L^2} = 1$. There exists a unique weak solution $\psi \in C^0([0,T], L^2(0,1))$ of*

$$\begin{cases} i\frac{\partial \psi}{\partial t} = -\psi'' - u(\psi)\mu\psi, x \in (0,1), t \in (0,+\infty), \\ \psi(t,0) = \psi(t,1) = 0, \\ \psi(0) = \psi_0, \end{cases} \tag{6.2.1}$$

where $u(\psi)$ is defined by (6.1.2). Moreover, these solutions are continuous with respect to initial conditions in the following sense: for every $T > 0$, there exists $C = C(T) > 0$ such that, for every $\psi_0^1, \psi_0^2 \in L^2(0,1)$ with $\|\psi_0^1\|_{L^2} = \|\psi_0^2\|_{L^2} = 1$, we have

$$\|\psi^1 - \psi^2\|_{L^\infty((0,T),L^2)} \leqslant C(T)\|\psi_0^1 - \psi_0^2\|_{L^2}. \tag{6.2.2}$$

Proof of Proposition 12: Let $T > 0$ be such that

$$2TC_\mu^2 e^{TC_\mu^2} < 1. \tag{6.2.3}$$

Let $\psi_0 \in L^2(0,1)$ be such that $\|\psi_0\|_{L^2} = 1$. We introduce the map

$$\begin{array}{rccc} F: & \overline{B}[C^0([0,T],L^2)] & \to & \overline{B}[C^0([0,T],L^2)] \\ & \xi & \mapsto & \psi \end{array}$$

where ψ solves

$$\begin{cases} i\frac{\partial \psi}{\partial t} = -\psi'' - u(\xi)\mu\psi, x \in (0,1), t \in (0,+\infty), \\ \psi(t,0) = \psi(t,1) = 0, \\ \psi(0) = \psi_0, \end{cases} \tag{6.2.4}$$

and $\overline{B}[C^0([0,T],L^2)] := \{\xi \in C^0([0,T],L^2); \|\xi(t)\|_{L^2} \leqslant 1, \forall t \in [0,T]\}$.

First, let us prove that F takes values in $\overline{B}[C^0([0,T],L^2)]$. For $\xi \in \overline{B}[C^0([0,T],L^2)]$, the function $u(\xi)$ is continuous over $[0,T]$, in particular, $u(\xi) \in L^1((0,T),\mathbb{R})$. Thus, Proposition 3 ensures the existence of a unique weak solution $\psi \in C^0([0,T],L^2)$ of (6.2.4); moreover, $\|\psi(t)\|_{L^2} = 1, \forall t \in [0,T]$, thus, $\psi \in \overline{B}[C^0([0,T],L^2)]$.

In order to prove Proposition 12 it is sufficient to prove that F has a fixed point. Thus, we apply the Banach fixed point theorem.

$(\overline{B}[C^0([0,T],L^2)], \|.\|_{L^\infty((0,T),L^2)}$ is a Banach space because it is a closed subset of the Banach space $(C^0([0,T],L^2), \|.\|_{L^\infty((0,T),L^2)})$.

For $\xi^1, \xi^2 \in \overline{B}[C^0([0,T],L^2)]$, we have $F(\xi^1) - F(\xi^2) = \psi^1 - \psi^2$ where

$$\begin{cases} i\frac{\partial(\psi^1-\psi^2)}{\partial t} = -(\psi^1-\psi^2)'' - u(\xi^1)\mu(\psi^1-\psi^2) - [u(\xi^1) - u(\xi^2)]\mu\psi^2, \\ (\psi^1-\psi^2)(t,0) = (\psi^1-\psi^2)(t,1) = 0, \\ (\psi^1-\psi^2)(0,x) = 0. \end{cases}$$

Applying Proposition 12, we get

$$\begin{aligned}\|\psi^1 - \psi^2\|_{L^\infty((0,T),L^2)} &\leqslant \|[u(\xi^1) - u(\xi^2)]\mu\psi^2\|_{L^1((0,T),L^2)} e^{C_\mu \|u(\xi^1)\|_{L^1(0,T)}} \\ &\leqslant TC_\mu \|u(\xi^1) - u(\xi^2)\|_{L^\infty(0,T)} e^{TC_\mu \|u(\xi^1)\|_{L^\infty(0,T)}}.\end{aligned}$$

Thanks to Cauchy-Schwarz inequality, we have

$$|u[\xi^1(t)]| = |\Im(\langle \mu\xi^1(t), \varphi_1\rangle \langle \varphi_1, \xi^1(t)\rangle)| \leqslant C_\mu,$$

$$\begin{aligned}|u(\xi^1) - u(\xi^2)| &\leqslant |\Im(\langle \mu\xi^1 - \xi^2, \varphi_1\rangle\langle\varphi_1, \xi^1\rangle)| + |\Im(\langle \mu\xi^2, \varphi_1\rangle\langle\varphi_1, \xi^1 - \xi^2\rangle)| \\ &\leqslant 2C_\mu \|\xi_1 - \xi_2\|_{L^2}\end{aligned}$$

thus,

$$\|\psi^1 - \psi^2\|_{L^\infty((0,T),L^2)} \leqslant 2TC_\mu^2 \|\xi_1 - \xi_2\|_{L^2} e^{TC_\mu^2}.$$

Thanks to (6.2.3), F is a contraction. □

6.3 Convergence: the non pathological case

The goal of this section is the proof of the following result.

Theorem 11 *We assume*

$$\langle \mu\varphi_1, \varphi_k\rangle \neq 0, \forall k \geqslant 2. \tag{6.3.1}$$

Let $\psi_0 \in H^2_{(0)}(0,1)$ *be such that* $\|\psi_0\|_{L^2} = 1$, $\langle \psi_0, \varphi_1\rangle \neq 0$ *and*

$$\begin{aligned}&\exists (t_n)_{n\in\mathbb{N}} \in [0,+\infty)^{\mathbb{N}} \text{ and } \xi_0 \in L^2(0,1) \text{ such that } t_n \to +\infty \\ &\text{and } \|\psi(t_n) - \xi_0\|_{L^2} \to 0, \text{ when } n \to +\infty.\end{aligned} \tag{6.3.2}$$

Then, the solution of (6.2.1) satisfies

$$\mathcal{L}(\psi(t)) \to 1 \text{ when } t \to +\infty.$$

As in the previous chapter, Theorem 11 will be proved thanks to LaSalle invariance principle. Let us emphasise that Theorem 11 contains the additional assumption (6.3.2) with respect to Theorem 10. This assumption is a compactness assumption (it corresponds more or less to the relative compactness of $\{\psi(t); t \in [0,+\infty)\}$ in $L^2(0,1)$). As we will see in the proof, this assumption is necessary to apply the LaSalle invariance principle. However, this assumption is also very difficult to check in general. In particular, the validity of (6.3.2) for the closed loop system (6.2.1) is still an open problem. Therefore, the asymptotic behaviour of (6.2.1) is still an open problem.

In order to prove Theorem 11, we first focus on the invariant set.

Proposition 13 *We assume (6.3.1) Let* $\psi_0 \in L^2(0,1)$ *with* $\|\psi_0\|_{L^2} = 1$*,* $\langle \psi_0, \varphi_1 \rangle \neq 0$*, and* ψ *be the solution of (6.2.1). We assume that* $t \mapsto \mathcal{L}(\psi(t))$ *is constant. Then, there exists* $\theta \in [0, 2\pi)$ *such that* $\psi(t) = e^{i\theta}\psi_1(t)$*.*

Proof of Proposition 13: From (6.1.3), we get $u(\psi) \equiv 0$. We have

$$i\frac{\partial \psi}{\partial t}(t,x) = -\psi''(t,x),$$

thus

$$\psi(t) = \sum_{k=1}^{\infty} \langle \psi_0, \varphi_k \rangle e^{-i\lambda_k t}\varphi_k.$$

From (6.1.2), we get

$$\Im\Big(\sum_{k=1}^{\infty} \langle \psi_0, \varphi_k \rangle \langle \mu\varphi_k, \varphi_1 \rangle e^{-i\lambda_k t}\overline{\langle \psi_0, \varphi_1 \rangle} e^{i\lambda_1 t}\Big) = 0, \forall t \in [0, +\infty),$$

or, equivalently,

$$\sum_{|k|=2}^{\infty} d_k e^{-i\omega_k t} = 0, \forall t \in [0, +\infty).$$

where $d_k := \langle \mu\varphi_k, \varphi_1 \rangle \langle \psi_0, \varphi_k \rangle \overline{\langle \psi_0, \varphi_1 \rangle}$, $d_{-k} := \overline{d_k}$, $\omega_k := \lambda_k - \lambda_1$, $\omega_{-k} := -\omega_k$, $\forall k \geqslant 2$. Therefore (see Theorem 7) $d_k = 0$ for every $k \geqslant 2$. Thanks to (6.3.1), we have $\langle \psi_0, \varphi_k \rangle \overline{\langle \psi_0, \varphi_1 \rangle} = 0$ for every $k \geqslant 2$. Since $\langle \psi_0, \varphi_1 \rangle \neq 0$, then $\langle \psi_0, \varphi_k \rangle = 0$ for every $k \geqslant 2$. Since $\|\psi_0\|_{L^2} = 1$, we conclude that $\psi(t) = e^{i\theta}\psi_1(t)$, for some $\theta \in [0, 2\pi)$. □

Proof of Theorem 11: We consider $\psi_0 \in L^2(0,1)$ be such that $\|\psi_0\|_{L^2} = 1$, $\langle \psi_0, \varphi_1 \rangle \neq 0$ and (6.3.2) holds.

The function $t \mapsto \mathcal{L}(\psi(t))$ is not decreasing, thus, there exists $L \in [\mathcal{L}(\psi_0), 1] \subset (0, 1]$ such that

$$\lim_{t \to +\infty} \mathcal{L}(\psi(t)) = L.$$

Let ξ be the solution of the closed loop system with initial condition ξ_0. Thanks to (6.2.2), we have $\psi(t_n + t) \to \xi(t)$, strongly in $L^2(0,1)$, when $n \to +\infty$, for every $t \in [0, +\infty)$. Thus

$$\mathcal{L}[\xi(t)] = \lim_{n \to +\infty} \mathcal{L}[\psi(t_n + t)] = L, \forall t \in [0, +\infty).$$

The map $t \mapsto \mathcal{L}[\xi(t)]$ is constant. Moreover, we have $\langle \xi_0^\infty, \varphi_1 \rangle \neq 0$ because $\mathcal{L}[\xi_0] = |\langle \xi_0, \varphi_1 \rangle|^2 = L \in (0, 1]$, and $\|\xi_0\|_{L^2} = 1$ thanks to (6.3.2). Thus, applying Proposition 13, we conclude that $\xi_0 = e^{i\theta}\varphi_1$ for some $\theta \in [0, 2\pi)$ and $L = 1$.□

Remark 8 *The compactness assumption (6.3.2) is needed in the previous proof. Indeed, let us assume that (6.3.2) is not satisfied and let us try to perform the same proof.*

We consider $(t_n)_{n\in\mathbb{N}} \in [0,+\infty)^{\mathbb{N}}$ *such that* $t_n \to +\infty$. *The sequence* $(\psi(t_n))_{n\in\mathbb{N}}$ *is bounded in* $L^2(0,1)$ *because* $\|\psi(t)\|_{L^2} = 1, \forall t \geqslant 0$. *Thus, there exists* $\xi_0 \in L^2(0,1)$ *such that* $\psi(t_n) \rightharpoonup \xi_0$ ***weakly*** *in* $L^2(0,1)$. *Then,* $\mathcal{L}(\xi_0) = \lim_{n\to+\infty} \mathcal{L}(\psi(t_n)) = L$. *We have* $\langle \xi_0, \varphi_1\rangle \neq 0$ *because* $L > 0$ *and* $\xi_0 \neq e^{i\theta}\varphi_1, \forall\theta \in [0,2\pi)$ *because* $L < 1$. *However, we cannot apply Proposition 13, because* ξ_0 *may not satisfy* $\|\xi_0\|_{L^2} = 1$. *Indeed, the weak convergence in* $L^2(0,1)$, $\psi(t_n) \rightharpoonup \xi_0$, *only provides* $\|\xi_0\|_{L^2} \leqslant \liminf_{n\to+\infty} \|\psi(t_n)\|_{L^2} = 1$ *and the inequality may be strict.*

6.4 An alternative: Approximate stabilisation

The goal of this section is to propose an adaptation of the strategy presented in the previous section, in order to be able to conclude without the compactness assumption (6.3.2). In order to do that, we introduce another Lyapunov function, which aims at controlling the growth of the hight frequencies (which are responsible for a possible non compactness).

6.4.1 Heuristic

For $N \in \mathbb{N}^*$ and $\epsilon \in (0,1)$, we introduce

$$\mathcal{L}_{N,\epsilon}(\psi) := 1 - |\langle\psi,\varphi_1\rangle|^2 - (1-\epsilon)\sum_{k=2}^{N} |\langle\psi,\varphi_k\rangle|^2. \tag{6.4.1}$$

This Lyapunov function encodes two tasks

- it prevents the L^2-mass lost through the high frequencies,
- it privileges the increase of the population in the ground state.

When ψ solves (1.1.1), we have

$$\frac{d\mathcal{L}_{N,\epsilon}(\psi)}{dt} = 2\sum_{k=1}^{N} a_k \Im\Big(\langle\mu\psi,\varphi_k\rangle\langle\varphi_k,\psi\rangle\Big)$$

where

$$a_k := \begin{cases} 1 \text{ if } k = 1, \\ (1-\epsilon) \text{ if } k \in \{2,\dots,N\}. \end{cases}$$

Thus, we consider the following feedback law

$$u(\psi) := -\sum_{k=1}^{N} a_k \Im\Big(\langle\mu\psi,\varphi_k\rangle\langle\varphi_k,\psi\rangle\Big), \tag{6.4.2}$$

which realizes

$$\frac{d\mathcal{L}_{N,\epsilon}(\psi)}{dt} = -u(\psi)^2 \leqslant 0. \tag{6.4.3}$$

6.4.2 Well posedness of the closed loop system

The well posedness of the closed loop system can be proved exactly as Proposition 12.

Proposition 14 *Let $\psi_0 \in L^2(0,1)$ be such that $\|\psi_0\|_{L^2} = 1$. There exists a unique weak solution of (6.2.1) where $u(\psi)$ is defined by (6.4.2). Moreover, these solutions are continuous with respect to the initial conditions for the $C^0([0,T], H^{-1}(0,1))$-topology: for every $T > 0$, there exists $C = C(T) > 0$ such that, for every $\psi_0^1, \psi_0^2 \in L^2(0,1)$ with $\|\psi_0^1\|_{L^2} = \|\psi_0^2\|_{L^2} = 1$, we have*

$$\|\psi^1 - \psi^2\|_{L^\infty((0,T),H^{-1})} \leqslant C(T)\|\psi_0^1 - \psi_0^2\|_{H^{-1}}. \tag{6.4.4}$$

Proof of (6.4.4): We have

$$\begin{aligned}&(\psi^1 - \psi^2)(t) = e^{-iAt}(\psi_0^1 - \psi_0^2)\\ &+i\int_0^T e^{-iA(t-s)}\Big(u(\psi^1(s))\mu(\psi^1 - \psi^2)(s) + [u(\psi^1(s)) - u(\psi^2(s))]\mu\psi^2(s)\Big)ds.\end{aligned}$$

For every $t \in \mathbb{R}$, e^{-iAt} is an isometry of H^{-1} for the norm

$$f \in H^{-1}(0,1) \mapsto \left(\sum_{k=1}^{\infty} \frac{1}{\lambda_k}|\langle f, \varphi_k\rangle|^2\right)^{1/2},$$

which is equivalent to the usual $H^{-1}(0,1)$-norm (i.e. $\|f\|_{H^{-1}} = \sup\{\langle f, \varphi\rangle; \varphi \in H_0^1(0,1)\}$). Thus, there exists $C_1 > 0$ such that

$$\|e^{-iAt}f\|_{H^{-1}} \leqslant C_1\|f\|_{H^{-1}}, \forall f \in H^{-1}(0,1).$$

Let C_μ be a positive constant such that

$$\|\mu f\|_{L^2} \leqslant C_\mu\|f\|_{L^2}, \forall f \in L^2(0,1),$$
$$\|\mu f\|_{H^{-1}} \leqslant C_\mu\|f\|_{H^{-1}}, \forall f \in H^{-1}(0,1),$$

(such a constant exists because, there exists $C'_\mu > 0$ such that, for every $\varphi \in H_0^1(0,1)$, $\|\mu\varphi\|_{H_0^1} = \|(\mu\varphi_1)'\|_{L^2} \leqslant C'_\mu\|\varphi\|_{H_0^1}$). Since $\|u(\psi^1)\|_{L^\infty} \leqslant C_\mu$, we have

$$\begin{aligned}&\|(\psi^1 - \psi^2)(t)\|_{H^{-1}} \leqslant C_1\|\psi_0^1 - \psi_0^2\|_{H^{-1}}\\ &+\int_0^t C_1\Big(C_\mu^2\|(\psi^1 - \psi^2)(s)\|_{H^{-1}} + |u(\psi^1(s)) - u(\psi^2(s))|C_\mu\Big)ds.\end{aligned}$$

We also have

$$\begin{aligned}|u(\psi^1) - u(\psi^2)| &= \Big|\Im\Big(\langle\mu(\psi^1 - \psi^2), \varphi_1\rangle\langle\varphi_1, \psi^1\rangle + \langle\mu\psi^2, \varphi_1\rangle\langle\varphi_1, \psi^1 - \psi^2\rangle\Big)\Big|\\ &\leqslant \|\psi^1 - \psi^2\|_{H^{-1}}\|\mu\varphi_1\|_{H_0^1} + \|\mu\psi^2\|_{L^2}\|\varphi_1\|_{H_0^1}\|\psi^1 - \psi^2\|_{H^{-1}}\\ &\leqslant C_2\|\psi^1 - \psi^2\|_{H^{-1}}\end{aligned}$$

where $C_2 = C_2(\mu, \varphi_1)$. Thus,

$$\|(\psi^1-\psi^2)(t)\|_{H^{-1}} \leqslant C_1\|\psi_0^1-\psi_0^2\|_{H^{-1}}+\int_0^t C_1[C_\mu^2+C_2C_\mu]\|(\psi^1-\psi^2)(s)\|_{H^{-1}}ds$$

Gronwall lemma gives (6.4.4) with $C(T) = C_1 e^{T[C_\mu^2+C_2C_\mu]}$.□

6.4.3 Convergence result in the non pathological case

Theorem 12 *We assume (6.3.1) and* $N = 3$. *Let* $\epsilon > 0$, $\gamma > 0$, $\psi_0 \in L^2(0,1)$ *be such that* $\|\psi_0\|_{L^2} = 1$,

$$|\langle\psi_0, \varphi_1\rangle| \geqslant \gamma, \tag{6.4.5}$$

$$\sum_{k=N+1}^{\infty} |\langle\psi_0, \varphi_k\rangle|^2 \leqslant \frac{\epsilon\gamma^2}{1-\epsilon}. \tag{6.4.6}$$

Then, the solution of the closed loop system (6.2.1), (6.4.2) satisfies

$$\liminf_{t\to+\infty} |\langle\psi(t), \varphi_1\rangle|^2 \geqslant 1-\epsilon. \tag{6.4.7}$$

Remark 9 *For every* $t \in [0,+\infty)$ *and* $\theta \in [0, 2\pi)$, *we have*

$$\begin{aligned} \|\psi(t) - \varphi_1 e^{i\theta}\|_{L^2}^2 &= \|\psi(t)\|_{L^2}^2 + \|\varphi_1 e^{i\theta}\|_{L^2}^2 - 2\Re(\langle\psi(t), \varphi_1 e^{i\theta}\rangle) \\ &= 2\Big(1 - \Re(e^{-i\theta}\langle\psi(t), \varphi_1\rangle)\Big) \\ &\leqslant 2\Big(1 - |\langle\psi(t), \varphi_1\rangle|\Big). \end{aligned}$$

Thus, (6.4.7) ensures that

$$\liminf_{t\to+\infty} dist_{L^2(0,1)}(\psi(t), \mathcal{C}_1) \leqslant 2(1-\sqrt{1-\epsilon}),$$

where $\mathcal{C}_1 := \{\varphi_1 e^{i\theta}; \theta \in [0, 2\pi)\}$, *i.e. the asymptotic distance to the ground state is arbitrarily small. Thus, the conclusion provides a strong convergence in* $L^2(0,1)$ *to the ground state.*

Remark 10 *The assumption (6.4.6) means that the main components of* ψ_0 *are on* φ_1, φ_2 *and* φ_3. *Notice that* ψ_0 *may be far from* $\mathcal{C}_1$ *(for example, when* γ *is small), thus the feedback law allows to go closer to* $\mathcal{C}_1$.

Remark 11 *In Theorem 12, we take* $N = 3$. *This assumption is useful in the proof, because one works on the particular model (1.1.1). However, for slightly different models (for example, with* ∂_x^2 *replaced by* $\partial_x^2 - \gamma\mu$ *in the Schrödinger equation),* N *may be chosen arbitrarily. We refer to [32], in which a global stabilisation result is proved, with the strategy of this course, on different models, that allow to take arbitrarily large* N.

The proof of Theorem 12 needs the following technical result.

Proposition 15 *Let* $k_1, j_1 \in \{1,2,3\}$ *and* $k_2, j_2 \in \mathbb{N}^*$ *be such that* $k_1 \neq k_2$, $j_1 \neq j_2$ *and* $\lambda_{k_1} - \lambda_{k_2} = \lambda_{j_1} - \lambda_{j_2}$. *Then* $(k_1, k_2) = (j_1, j_2)$.

Proof of Proposition 15: The conclusion is obvious when $k_1 = j_1$, thus, it is sufficient to treat the following cases $(k_1, j_1) = (1, 2)$, $(k_1, j_1) = (1, 3)$, $(k_1, j_1) = (2, 3)$.

Let $k > 1$, $j > 2$ be such that $k^2 - 1 = j^2 - 4$. Then $j^2 - k^2 = (j+k)(j-k) = 3$, so $j + k = 3$ and $j - k = 1$, i.e. $j = 2$ and $k = 1$: contradiction.

Let $k > 1$, $j > 3$ be such that $k^2 - 1 = j^2 - 9$. Then $j^2 - k^2 = (j+k)(j-k) = 8$. The first possibility is $j + k = 8$ and $j - k = 1$, which leads to $j = 9/2$, $k = 7/2$: contradiction. The second possibility is $j + k = 4$ and $j - k = 2$, which leads to $j = 3$, $k = 1$: contradiction.

Let $k > 2$, $j > 3$ be such that $k^2 - 4 = j^2 - 9$. Then $j^2 - k^2 = (j+k)(j-k) = 5$, so $j + k = 5$ and $j - k = 1$, i.e. $j = 3$, $k = 2$: contradiction. $\square$

Remark 12 *Notice that Proposition 15 is false if we consider* $k_1, j_1 \in \{1,2,3,4\}$. *Indeed, we have* $\lambda_7 - \lambda_1 = \lambda_8 - \lambda_4$.

Proof of Theorem 12: The function $t \mapsto \mathcal{L}_{N,\epsilon}(\psi(t))$ is not increasing, thus, there exists $L \in [0, \mathcal{L}_{N,\epsilon}(\psi_0)]$ such that $\mathcal{L}_{N,\epsilon}(\psi(t)) \to L$ when $t \to +\infty$. Using $\|\psi_0\|_{L^2} = 1$, (6.4.5) and (6.4.6), we get

$$\begin{aligned}
\mathcal{L}(\psi_0) &= 1 - \epsilon |\langle \psi_0, \varphi_1 \rangle|^2 - (1-\epsilon) \textstyle\sum_{k=1}^{N} |\langle \psi_0, \varphi_k \rangle|^2 \\
&= 1 - \epsilon |\langle \psi_0, \varphi_1 \rangle|^2 - (1-\epsilon) \Big(1 - \textstyle\sum_{k=N+1}^{\infty} |\langle \psi_0, \varphi_k \rangle|^2 \Big) \\
&< 1 - \epsilon \gamma^2 + (1-\epsilon) \Big(1 - \tfrac{\epsilon \gamma^2}{1-\epsilon} \Big) \\
&= \epsilon.
\end{aligned}$$

Thus, $L \in [0, \epsilon)$. Let $(t_n)_{n \in \mathbb{N}} \in [0, +\infty)^{\mathbb{N}}$ be such that $t_n \to +\infty$ when $n \to +\infty$. The sequence $(\psi(t_n))_{n \in \mathbb{N}}$ is bounded in $L^2(0,1)$, thus, there exists $\xi_0 \in L^2(0,1)$ and an extraction s such that $\psi(t_{s(n)}) \to \xi_0$ weakly in $L^2(0,1)$ and strongly in $H^{-1}(0,1)$. Then, thanks to (6.4.4), $\psi(t_{s(n)} + T) \to \xi(T)$ strongly in $H^{-1}(0,1)$, for every $T > 0$. Thus $\mathcal{L}(\xi(t)) = L, \forall t \in [0, +\infty)$. From (6.4.3), we deduce that $u(\xi) \equiv 0$ and

$$\xi(t) = \sum_{j=1}^{\infty} \langle \xi_0, \varphi_j \rangle e^{-i\lambda_j t} \varphi_j.$$

The equality $u(\xi) = 0$ gives

$$\Im \Big(\sum_{k=1}^{N} \sum_{j=1}^{\infty} \langle \xi_0, \varphi_j \rangle e^{-i\lambda_j t} \langle \mu \varphi_j, \varphi_k \rangle \langle \varphi_k, \xi_0 \rangle e^{i\lambda_k t} \Big) \equiv 0.$$

Thanks to Theorem 8, and Proposition 15, we get

$$\langle \xi_0, \varphi_j \rangle \langle \mu\varphi_j, \varphi_1 \rangle \langle \varphi_1, \xi_0 \rangle = 0, \forall j \geqslant 2.$$

Thanks to (6.3.1), we get

$$\langle \xi_0, \varphi_k \rangle \langle \varphi_1, \xi_0 \rangle = 0, \forall j \geqslant 2. \tag{6.4.8}$$

Let us prove that $\langle \varphi_1, \xi_0 \rangle \neq 0$. Since $\|\xi_0\|_{L^2} = 1$, we have

$$\begin{array}{ll} \epsilon & > \mathcal{L}(\xi_0) \\ & = 1 - \epsilon |\langle \xi_0, \varphi_1 \rangle|^2 - (1-\epsilon) \sum_{k=1}^{N} |\langle \xi_0, \varphi_1 \rangle|^2 \\ & \geqslant 1 - \epsilon |\langle \xi_0, \varphi_1 \rangle|^2 - (1-\epsilon) \end{array}$$

i.e. $|\langle \xi_0, \varphi_1 \rangle| > 0$. Thus, we deduce from (6.4.8) that $\xi_0 = z\varphi_1$ for some $z \in \mathbb{C}$ with $|z| \leqslant 1$. Then, $\mathcal{L}(\xi_0) = 1 - |z|^2 < \epsilon$, thus

$$\lim_{n \to +\infty} |\langle \psi(t_{s(n)}), \varphi_1 \rangle|^2 = |z|^2 > 1 - \epsilon.$$

This holds for any sequence $(t_n)_{n \in \mathbb{N}}$, thus we have (6.4.7).$\square$

6.4.4 Convergence result in the pathological case

We refer to [32] for a convergence result when the assumption (6.3.1) is not satisfied.

.1 Cauchy-Lipschitz theorem, Gronwall Lemma

.2 Vandermonde determinant

Proposition 16 *Let $N \in \mathbb{N}^*$ and $a_1, ..., a_N \in \mathbb{R}$. We have*

$$\begin{vmatrix} 1 & a_1 & a_1^2 & ... & a_1^{N-1} \\ 1 & a_2 & a_2^2 & ... & a_2^{N-1} \\ ... & ... & ... & ... & ... \\ 1 & a_N & a_N^2 & ... & a_N^{N-1} \end{vmatrix} = \Pi_{1 \leqslant i < j \leqslant N}(a_j - a_i).$$

Proof of Proposition 16: We prove Proposition 16 by induction on N. The result is clear for $N = 2$. Let $N > 2$ and let us assume that it is proved for $N - 1$. Replacing the j^{th} column C_j by $C_j - a_1 C_{j-1}$ for $j = N, ..., 2$ (starting from the N^{th} one) we get

$$\begin{aligned} V(a_1, ..., a_n) &= \begin{vmatrix} 1 & 0 & 0 & ... & 0 \\ 1 & (a_2 - a_1) & (a_2 - a_1)a_2 & ... & (a_2 - a_1)a_2^{N-2} \\ ... & ... & ... & ... & ... \\ 1 & (a_N - a_1) & (a_N - a_1)a_2 & ... & (a_N - a_1)a_2^{N-2} \end{vmatrix} \\ &= \begin{vmatrix} (a_2 - a_1) & (a_2 - a_1)a_2 & ... & (a_2 - a_1)a_2^{N-2} \\ ... & ... & ... & ... \\ (a_N - a_1) & (a_N - a_1)a_N & ... & (a_N - a_1)a_N^{N-2} \end{vmatrix} \\ &= (a_2 - a_1)...(a_N - a_1) \begin{vmatrix} 1 & a_2 & ... & a_2^{N-2} \\ ... & ... & ... & ... \\ 1 & a_N & ... & a_N^{N-2} \end{vmatrix} \\ &= (a_2 - a_1)...(a_N - a_1)V(a_2, ..., a_N). \square \end{aligned}$$

Corollary 1 *Let $N \in \mathbb{N}^*$, $T > 0$ and $a_1, ..., a_N \in \mathbb{R}$ be such that $a_1 < ... < a_N$. Then the family $(e^{ia_1 t}, ..., e^{ia_N t})$ is linearly independent in $L^2((0,T),\mathbb{C})$. Thus its Gram matrix for the $L^2((0,T),\mathbb{C})$ scalar product, $G = (G_{k,l})_{1 \leqslant k,l \leqslant N}$,*

$$G_{k,l} := \int_0^T e^{i(a_k - a_l)t} dt, \forall k, l \in \{1, ..., N\}$$

is invertible.

Proof of Corollary 1: Let $\alpha_1, ..., \alpha_N \in \mathbb{C}$ be such that $\alpha_1 e^{ia_1 t} + ... + \alpha_N e^{ia_N t} = 0$ in $L^2(0,T)$. The lhs is a continuous function of t, thus, the equality holds for every $t \in [0,T]$. The lhs is a C^∞ function of t, thus, we can differentiate the relation k times with respect to t, for $k = 0, ..., N-1$, and evaluate it at $t = 0$. This gives $\alpha_1 a_1^k + ... + \alpha_N a_N^k = 0, \forall k \in \{0, ..., N-1\}$, i.e.

$$\begin{pmatrix} 1 & 1 & ... & 1 \\ a_1 & a_2 & ... & a_N \\ ... & ... & ... & ... \\ a_1^{N-1} & a_2^{N-1} & ... & a_N^{N-1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ ... \\ \alpha_N \end{pmatrix} = 0.$$

Thanks to Proposition 16, the matrix in the lhs is invertible. Therefore $\alpha_1 = ... = \alpha_N = 0$. We have proved that the family $(e^{ia_1 t}, ..., e^{ia_N t})$ is linearly independent in $L^2((0,T),\mathbb{C})$.

Let $\beta_1, ..., \beta_N \in \mathbb{C}$ be such that $\beta_1 C_1 + ... + \beta_N C_N = 0$, where C_j is the j^{th}-column of G. Then, we have

$$\int_0^T e^{ia_k t}\overline{\left(\sum_{l=1}^N \beta_l e^{ia_l t}\right)}dt = 0, \forall k \in \{1, ..., N\}.$$

The function $t \mapsto \sum_{l=1}^N \beta_l e^{ia_l t}$ belongs to the final dimensional space $\mathrm{Span}(e^{ia_1 t}, ..., e^{ia_N t})$ and it is orthogonal to any vector of its basis, thus this function vanishes. Since the family $(e^{ia_1 t}, ..., e^{ia_N t})$ is linearly independent, we conclude that $\beta_1 = ... = \beta_N = 0$. We have proved that the column of G are linearly independent in $\mathbb{C}^N$, thus, G is invertible. □

.3 Weak convergences

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